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No. 143.

THE INSTITUTION OF
POST OFFICE ELECTRICAL ENGINEERS

ELECTRIC WAVE FILTERS

BY

G. J. S. LITTLE, B.Sc.

A PAPER

Read before the London Centre of the Institution on the 8th December, 1931.

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ELECTRIC WAVE FILTERS.

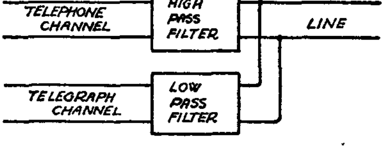
Electric Filters may be described as artificial lines composed of combinations of inductances and condensers so arranged that currents within certain bands of frequencies are attenuated more or less severely while other currents are transmitted with comparatively small losses. Some of the more important uses of filters are briefly referred to below.

Two-wire Repeaters.

The amount of amplification which can satisfactorily be obtained from a two-wire repeater is limited by the accuracy with which the balances can be made to simulate the impedances of the lines on either side of the repeater. Owing to slight irregularities in the capacities of sections of cable between loading coils, and in the inductance of individual loading coils, etc., the curves of impedance which have to be imitated by the balances are "bumpy." The departures from a mean smooth curve, which is all that it is practicable to reproduce in a balance, become more pronounced at the higher frequencies and it is desirable to suppress frequencies above about 2200 p.p.s. on medium-heavy loaded circuits by including low-pass filters in the circuits of the two-wire repeaters.

Sub-Audio Telegraphy.

For the satisfactory transmission of telephone speech, frequencies below 200 p.p.s. are not required and frequencies up to, say, 150 p.p.s. can be used for telegraph purposes by an arrangement of filters such as is indicated in the diagram.



The filters are used to prevent the telegraph currents causing interference in, or being unduly shunted by, the telephone apparatus, and vice-versa.

Problems presented by " Composited Telegraph and Telephone Working " were dealt with by Messrs. J. M. Owen and J. A. S. Martin in a paper read before this Institution in January, 1930.*

Voice-Frequency Telegraphy.

By means of Voice-Frequency Telegraphy the range of frequencies normally used in the transmission of speech can be utilised to provide a large number of telegraph channels. Signals are transmitted over each channel by keying a pure tone, which is rectified at the receiving end of the circuit to produce signals to actuate normal telegraph apparatus. By the use of tones spaced at intervals of 120 p.p.s. 18 duplex channels can be provided over a four-wire repeatered telephone circuit. Band-pass filters are normally employed in the voice-frequency portion of the circuit to prevent interference between the various channels at the terminals. Reference should be made to Mr. W. Cruickshank's paper on the subject of Voice Frequency Telegraphy read before this Institution in February, 1927.†

Carrier-Current Telephony.

Capt. A. C. Timmis, in his paper on " Carrier-Current Telephony,"‡ read before this Institution in February, 1930, showed the dependence of carrier-current telephone circuits upon filters for the prevention of interference between channels of communication provided by the carrier system and also between them and the direct speech circuit on which the system is superposed.

Filters, no doubt, are being increasingly used in laboratory work, and particularly in research work. Visitors to the recent Faraday Centenary Exhibition had the opportunity of hearing an effective demonstration, arranged by the P.O. Engineering Research Section, of the effect of cutting out, from speech and music, various portions of the frequency spectrum. This was accomplished by the use of low-pass and high-pass filters.

In spite of a widespread notion that a knowledge of com-

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* Paper No. 130.

† Paper No. 113.

‡ Paper No. 131.

plicated mathmatics is needed for an understanding of the subject, the theory of electric filters, at least in regard to steady-state conditions, involves little more than algebra and vectorial notation.

The main object of this paper is to give a straightforward introduction to the subject which will include a presentation of the most important principles involved on the theoretical side of filter design. Attention will be directed to high-pass and low-pass filters, but the principles applied are equally valid in the cases of more complicated structures such as band-pass filters, which cannot be dealt with in the present paper.

Affinity of Loaded Line and Low-pass Filter.

The inability of a coil-loaded line to transmit efficiently frequencies in the neighbourhood of its " cut off " frequency and above is well known, and the close relationship which exists between a loaded line and a low-pass filter will be used by way of introduction.

Attenuation Characteristics of Loaded Lines and Low-pass Filters.

Fig. 1 illustrates this discrimination by a loaded line against the higher frequencies. The curves have been drawn for cable with 120 mH. loading coils at intervals of 2000 yards —a loading which is being used for new construction as affording the minimum grade of transmission, in regard to frequency range, required by the Comité Consultatif International. The curve marked A shows the attenuation of a 20 lb. pair, rising ever more steeply as the " cut off " frequency—about 3300 p.p.s., marked by the vertical line F— is approached. In electric wave filters one is concerned with attenuation per section and the curves in Fig. 1 show the attenuation in decibels, not per mile but for each 2000 yards, that is to say the attenuation per loading section. Curve C shows the corresponding attenuation per loading-coil section of similarly loaded 40 lb. conductors. Curves A and C have been calculated from average values of quad cable and of present-day main line loading coils by the Mayer formula.*

The attenuations of unloaded cable, per 2000 yards, are marked B (20 lb.) and D (40 lb.). The curves for the loaded cables turn down at the lower frequencies to avoid, as one

* See Bibliography 6.

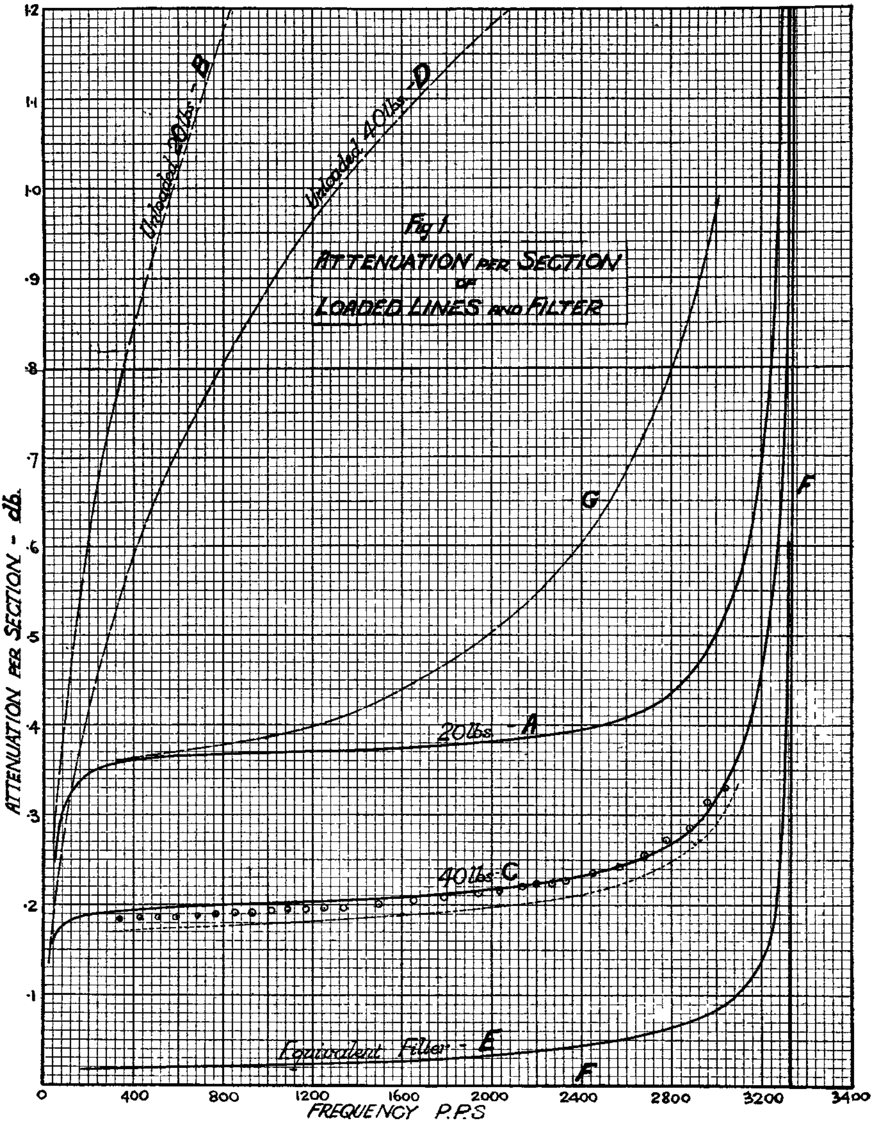
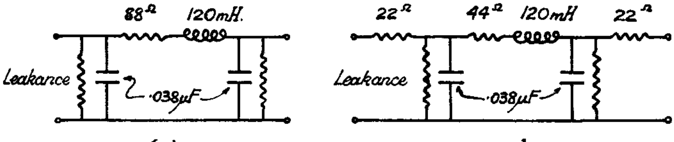


FIG. 1.

might almost say, crossing the attenuation curves for the unloaded cables.

The series of circles near curve C represents measurements of attenuation made by the Research Section on 40 lb. pairs in the Glasgow-Ayr (1931) cable. loaded with 120 mH coils at 2000 yards. The lightly dotted line is the attenuation curve calculated from the constants of that cable. The correspondence between the measured and calculated values is not as close as might perhaps have been expected, but the shapes of the curves are in good agreement. The author is indebted to Mr. W. T. Palmer for the results of these tests which were carried to a higher frequency than is usual in acceptance tests to check up on the Mayer formula.

Curve G shows the attenuation per section that one would get with an artificial line in which all the cable constants were lumped, as shown at (*a*). This attenuation-frequency curve is but a poor imitation of the loaded line. If the attenuation-frequency curve of loaded lines had been of this sort, the problem of distortion correction would indeed have been a



(*a*) (*b*)

serious one. By arranging the resistance as indicated at (*b*) no doubt a closer correspondence with the attenuation curve of the loaded line would be obtained, but it is perhaps worth noting that the resistance of the loading coils is more productive of slope in the attenuation-frequency curve than an equal " distributed " resistance, curve G illustrating the extreme case of resistance concentration.

Curve E is the attenuation per section of a low-pass filter having the same inductance (120 mH.) and capacity (.076 μF.) per section as each loading section of the loaded cable, (but no resistance except that of the loading coils). If the filter coils are assumed to have no resistance the attenuation-frequency curve is just a straight line at the bottom of the diagram from zero frequency as far as the " cut off " frequency when it suddenly shoots up almost vertically. The line marked F has been drawn to correspond with the theoretical attenuation for this case. The sudden rocket-like ascent of the curve supplies a justification for the term " cut off " frequency, but this diagram only extends to an attenua-

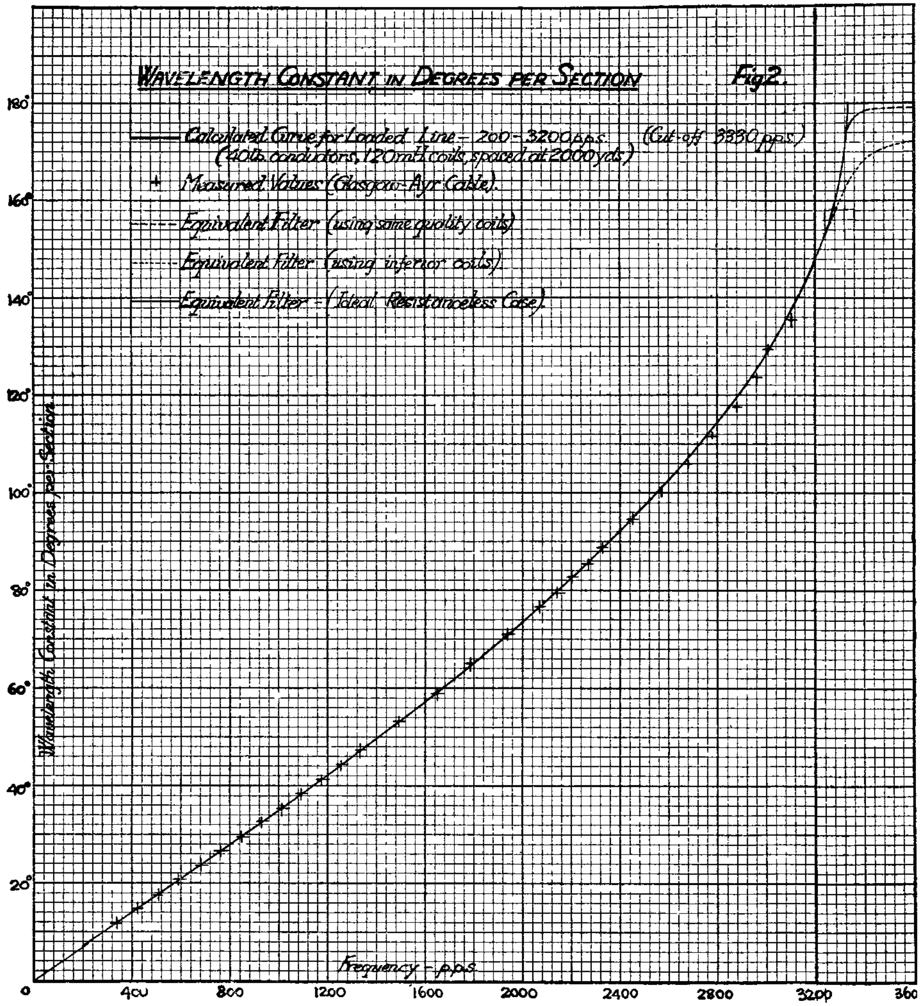


FIG. 2.

tion not much greater than one db. per section. When designing a filter to give an attenuation of say 40 db. one realises that the "cut off" frequency is the frequency at which attenuation theoretically begins, for, as will be shown in a later diagram, the attenuation-frequency curve does not continue to rise so rapidly, but bends away towards the right.

Phase-Change Curves.

Fig. 2 shows wave-length constant calculated in terms of degrees per loading coil section for the 40 lb. case. It is notable that, except at the extreme ends of the range of transmission, the phase change of the loaded line is practically identical with the corresponding resistanceless filter.

The calculated curve which is based upon the line constants of the Glasgow-Ayr cable, is in close agreement with measured values up to 2400 p.p.s. (70% of the cut off frequency), and even above this frequency the divergence is but small.

Characteristic Impedance—Half Section Temination.

Fig. 3 shows curves of characteristic impedance of the loaded line that is under consideration, plotted against a frequency base. The calculated curves refer to 20 lb. conductors and represent the real and imaginary components of the impedance that would be measured at the beginning of a long length of loaded line when the distance to the first loading coil is 1000 yards, that is, half a full loading section. The light broken curves were plotted from measurements in an actual case. Curves have also been drawn to indicate the impedance of the equivalent low-pass filter. The real part is practically identical with that of the loaded line except at the low frequencies, where the curve continues horizontal instead of turning up as in the case of the loaded line. The imaginary part of the impedance of the filter is so small in the middle of the range that it is only possible to indicate it satisfactorily on the diagram at the low frequencies and at frequencies near the cut off. The curve of the unreal component of the impedance of a line with 40 lb. conductors would lie about midway between those of the 20 lb. loaded line and the filter. There is a close relationship between the attenuation and the magnitude of the imaginary component of the characteristic impedance, and in an ideal filter with no resistance, the impedance would have no imaginary component. The impedance curve of unloaded 20 lb. conductors have been included in the diagram—it may be noted that the loaded and unloaded impedance curves are asymptotic to each other at low frequencies.

Half-Coil Termination.

Fig. 4 is a further set of impedance curves—these are for half-coil termination, that is to say, for a line commencing

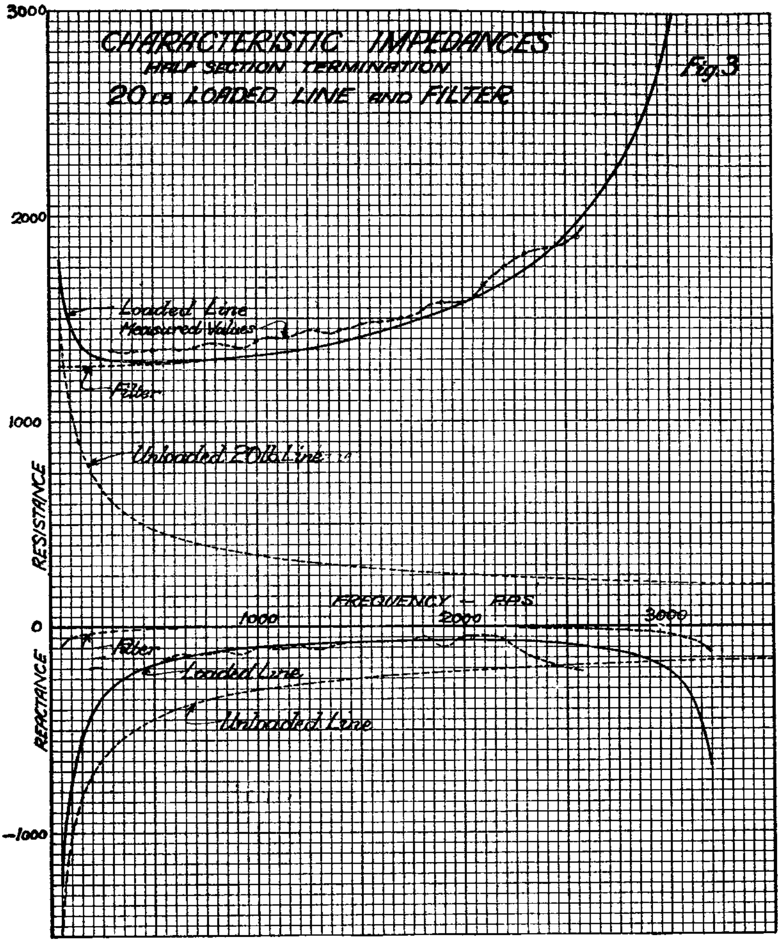


FIG. 3.

with a half value loading coil—60 mH.—followed by a full loading section. The impedance grows smaller as the frequency approaches the cut off, instead of rising as with the half-section termination. Measurements of impedance curves for half-coil termination in this loading are not available.

Half-Coil and Half-Section Terminations.

The two methods of terminating a loaded line, which as far as possible are used in practice, corresponding to the two

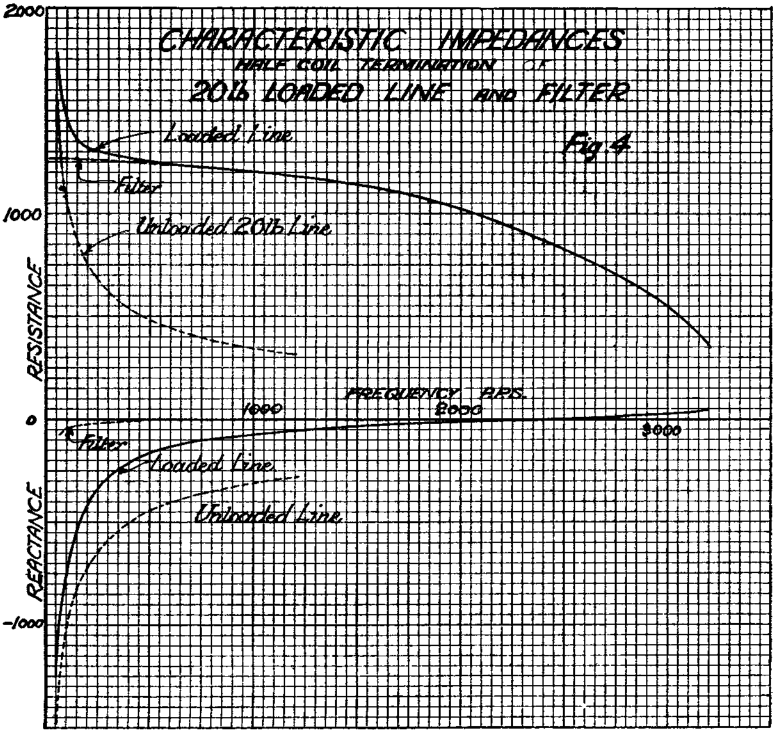


FIG. 4.

sets of impedance curves of Figs. 3 and 4, are illustrated in Fig. 5. When speaking of lines these are usually referred to as half-section termination and half-coil termination respectively, but it is usual to speak of a filter as having a mid-shunt or mid-series termination. When filters are terminated in either of these ways the characteristic impedance throughout the transmitting range—in the ideal resistanceless case—is non-reactive.

It is clear that, apart from the method of dealing with the end, the two lines are identical and, indeed, the mid-shunt and the mid-series impedances are equally characteristic of a filter (or of a loaded line).

Filter built up of "Half-Sections."

The lowest diagram, representing a filter terminated at mid-series, has been drawn split up into "half-sections,"

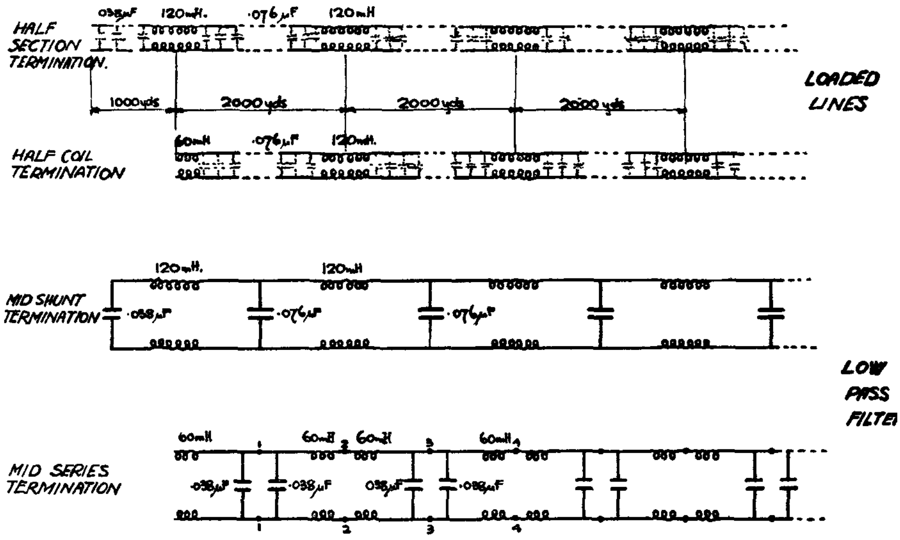


FIG. 5.

each of which are equal, but alternate " half-sections " are reversed.

Let it be supposed that the filter is continued to an indefinitely large number of sections to the right and that the left hand end is closed by the mid-series characteristic impedance of the filter. If the filter be cut at 11 the junction between two half-sections, looking towards the left and also to the right, the impedance seen will be the mid-shunt characteristic impedance. If the filter be cut at 22 it is the mid-series characteristic which will be presented and at successive junctions the two impedances will alternate.

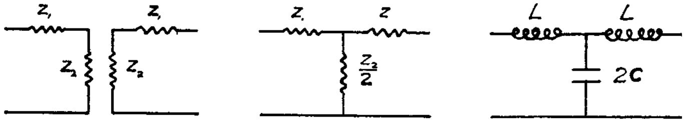
A filter such as is represented in the diagram can be conceived as being built up of half-sections placed " back to back." The half-section is the smallest part of such a structure which is representative of the whole and may be, perhaps not inaptly, described as a filter reduced to its lowest terms.

Balanced and Unbalanced Filters.

The diagrams of Fig. 5 show each inductance divided between the *a* and *b* wires of the filter. From a theoretical point of view the same effect would be obtained by concentrating all the inductance in, say, the *a* wire, the *b* wire then being without impedance. In practice the balanced arrangement, with division of the series impedances between the two wires, differs in its effect from the unbalanced type only in so far as the presence of capacities to earth are important.

General Formula for Characteristic Impedance.

Taking an unbalanced diagram, for simplicity, of two half-sections of a filter, shown at (*c*), and calling the impedance of series elements z_1 and of the shunt elements z_2, one can represent a whole mid-series section as at (*d*).



(c) (d) (e)

In Appendix I. it is shown that the mid-series characteristic impedance is given by the formula

$$Z_{01} = \sqrt{z_1 z_2 + z_1^2} \quad \ldots\ldots (1)$$

To make the diagram represent a low-pass filter section, it is necessary to substitute an inductance L for z_1 and a capacity C for z_2 as shown at (*e*). The impedances z_1 and z_2 are both imaginary, z_1 being positive $= j\omega L$ and Z_2 negative $= \frac{1}{j\omega C}$. The effect of this on the formula for characteristic impedance is to introduce a negative sign thus :—

$$Z_{01} = \sqrt{|z_1 z_2| - |z_1^2|} \quad \ldots\ldots (2)$$

So long as the magnitude of z_2 is greater than that of z_1 the quantity under the root is positive. When $|z_2| = |z_1|$ $Z_{01} = 0$ and when $|z_2| < |z_1|$ the quantity under the root is negative, Z_{01} necessarily being unreal in this case.

Cut Off Frequency.

The frequency which lies at the boundary between real and unreal characteristic impedance is the " cut off " frequency and, since this occurs when $|z_1| = |z_2|$, the formula for the " cut off " frequency of a low-pass filter is the same as that for the resonating frequency of a simple tuned circuit, *i.e.*,

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \qquad (3)$$

Formula for Mid-Series Impedance.

Substitution in equation (1) gives the formula

$$Z_{01} = \sqrt{\frac{L}{C} - \omega^2 L^2}$$

which can be written, with the aid of equation (3), in the form

$$Z_{01} = \omega_0 L \sqrt{1 - \left(\frac{\omega}{\omega_0}\right)^2} \qquad \qquad (4)$$

Nominal Impedance.

At low frequencies $Z_{01} = \omega_0 L$ and this is referred to as the nominal impedance of the filter and is designated by the letter K.

Characteristic Impedances of Low-Pass Filter.

Fig. 6 shows the curves of characteristic impedances of a resistanceless filter, whose cut off frequency is 1000 p.p.s., plotted against frequency. The inductances and capacities have been chosen to give a nominal impedance of 1000 ohms. It has been assumed that no resistance is associated with either inductances or condensers. As might be expected from an examination of the circuit the impedance in the mid-series case, after cut off, is asymptotic to the straight line through the origin representing the impedance of the inductance L. This line, at the cut off frequency, passes through a point corresponding to 1000 ohms, for $\omega_0 L = K$.

The corresponding curves for mid-shunt termination are also given, the equation being

$$Z_{02} = \frac{\omega_0 L}{\sqrt{1 - \left(\frac{\omega}{\omega_0}\right)^2}} \qquad \qquad (5)$$

The impedance, after cut off, approaches more and more closely to that of the terminating condenser, which is inversely proportional to the frequency, passing through $-1000j$ at the cut off frequency.

Propagation Characteristics.

The propagation constant γ for a symmetrical network, of π or T forms, is most readily calculated from the formula

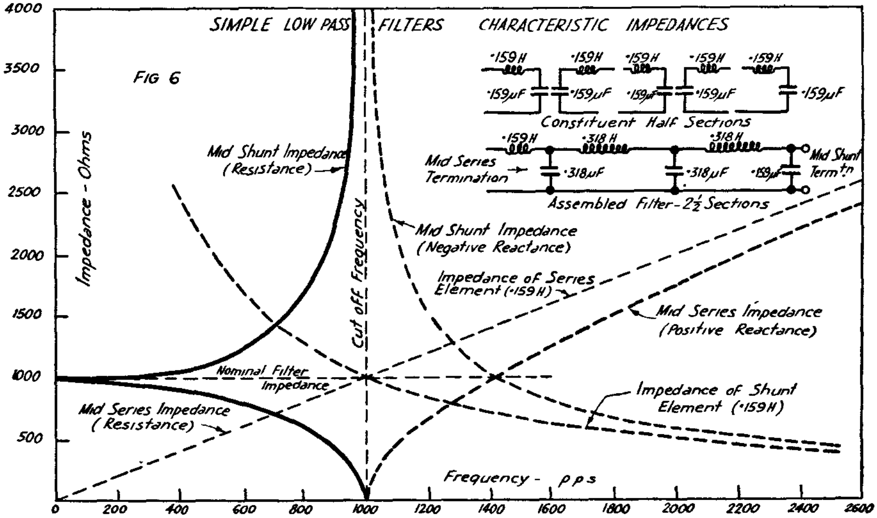


FIG. 6.

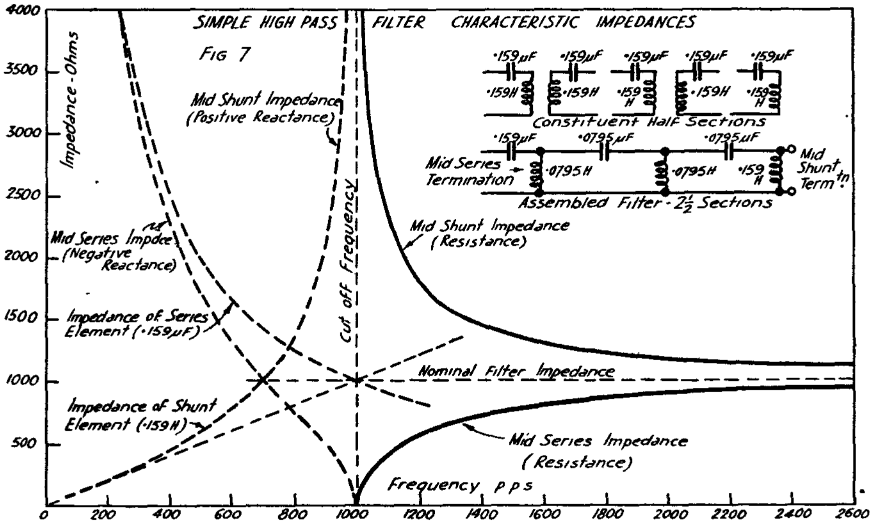


FIG. 7.

$$\cosh \gamma = 1 + \frac{2z_1}{z_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where z_1 and z_2 have the same significance as in formulæ previously given for characteristic impedance. Substitution of appropriate values for these quantities gives, for a complete section of a resistanceless low-pass filter, the formula

$$\cosh \gamma = 1 - 2\left(\frac{\omega}{\omega_0}\right)^2 \qquad \text{.....(7)}$$

Transmitting Range.

e^{γ} is the vector ratio of current (or volts) at the beginning of a section to current (or volts) at the end of the section, forming part of an infinite chain of identical sections. In the transmitting range in the resistanceless case this is equivalent to a change of phase only and the angular change of phase ϕ is given by the equation

$$\cosh \gamma = \cos \phi \qquad \text{.....(8)}$$

Attenuating Range.

In the attenuating range the magnitude of cosh γ is greater than unity and γ can be conveniently found from tables of hyperbolic functions and, if desired, expressed in decibels.

The attenuation and phase change per section for a low-pass filter, without resistance, are represented in Fig. 8. The curves for a filter in which the effective resistance of the inductances is $\frac{1}{20}$ of the reactance values are shown by broken lines where the curves are appreciably different from those of the resistanceless filter. This ratio of resistance to reactance represents approximately the power factor obtainable in a coil with a core formed of iron stampings with an air-gap. If the curves had been drawn for a ratio of 1 : 250, such as is obtainable with coils wound on dust cores, the curves would have been almost indistinguishable from the ideal resistanceless case except in the immediate neighbourhood of the cut off frequency. A ratio of 1 : 50 is obtainable with air-core coils of reasonable size.

Examination of the Mode of Action of a Filter.

At this stage an attempt may be made to throw some light on the action of a filter in transmitting a certain band of frequencies while discriminating against others. An electric filter is essentially a structure consisting of inductances and condensers which are incapable of absorbing power. In so far as the filter elements may be considered as being free from resistance any power which one can succeed in impress-

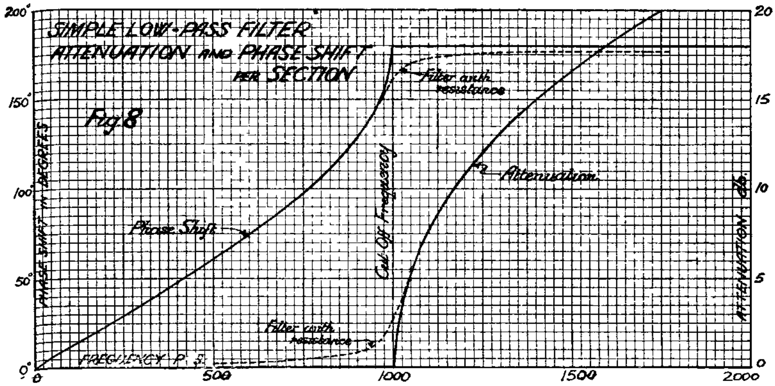

SIMPLE LOW-PASS FILTER
ATTENUATION AND PHASE SHIFT PER SECTION
Fig 8
Filter with resistance
Attenuation
Phase Shift
Cut Off Frequency
Filter with resistance
FREQUENCY P.S.
PHASE SHIFT IN DEGREES
ATTENUATION
200°
150°
100°
50°
0°
20
15
10
5
0
0
500
1000
1500
2000

FIG. 8.

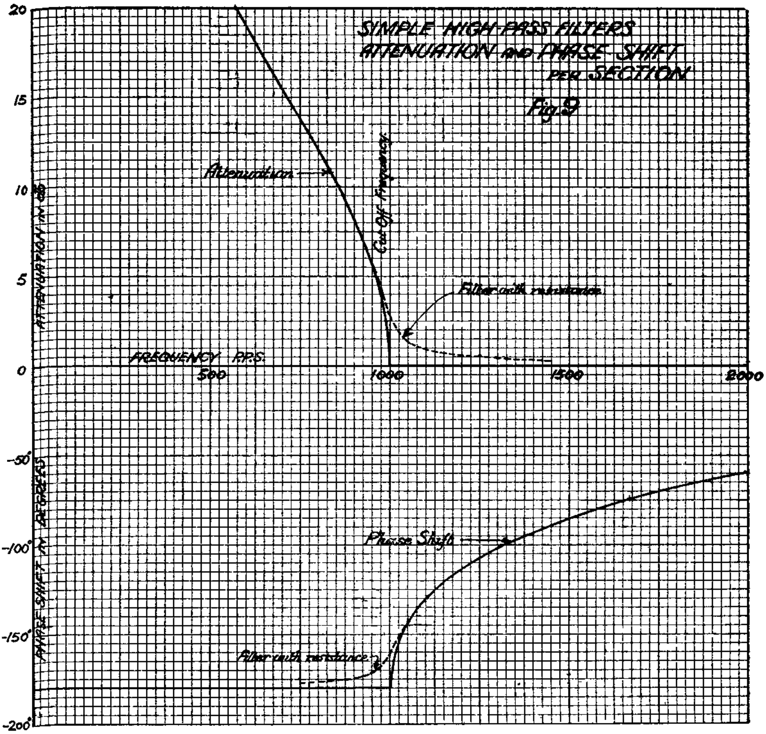

SIMPLE HIGH-PASS FILTERS
ATTENUATION AND PHASE SHIFT PER SECTION
Fig 9
Attenuation
Cut Off Frequency
Filter with resistance
FREQUENCY P.P.S.
Phase Shift
Filter with resistance
ATTENUATION IN
PHASE SHIFT IN DEGREES
20
15
10
5
0
-50°
-100°
-150°
-200°
500
1000
1500
2000

FIG. 9.

ing on the filter at its input terminals must of necessity be delivered without loss at its output terminals.

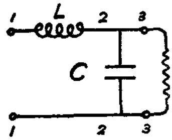
If the output terminals of the filter are closed by a resistance of the same order as the nominal impedance of the filter, the input impedance will be sufficiently non-reactive, up to within, say, 10 per cent. of the cut off frequency, to allow of efficient transfer of power from apparatus approximating in impedance to the nominal impedance of the filter.

In the attenuating range, coupled with an imaginary characteristic impedance, a filter consisting of several sections has, in general, a rapidly increasing attenuation constant, with the result that, at frequencies a few per cent. past the cut off, the input impedance is hardly affected by the impedance closing the output terminals. The input impedance is practically indistinguishable from the characteristic impedance of the filter and the smallness of the amount of power transmitted to the terminal apparatus throughout the attenuating range could be attributed to the imaginary nature of the input impedance.

It is true that attenuation of current and voltage does take place in a filter in the attenuating range, but, as in the ideal case, during their passage through the filter, current and voltage are 90° out of phase attenuation of power does not take place. The simple relation between the series and shunt elements which determines whether the ratio of currents (or volts) between the beginning and end of a section forming part of an infinite series of such sections shall be in the nature of attenuation or of change of phase only has been pointed out. This is a mathematical relation inseparable from the existence of impedances of opposite sign.

Interaction of Impedances of Filter Elements.

A brief explanation of the interaction between the series and shunt impedance elements of a filter in the transmitting range may be useful. Let the diagram at (*f*) represent an ideal half-section of a low-pass filter, terminated by a resistance equal to its nominal impedance K. At a low frequency, say, at a frequency which is $\frac{1}{n}$th of the cut off frequency f_0, the impedance of the capacity is high compared with K. Its value will be $-jn$K. (The impedance of the capacity C, and of the inductance L are both equal, in magnitude, to the



(f)

nominal impedance of the filter at the cut off frequency. Reference should be made to Fig. 6). The joint admittance of the condenser in parallel with K is

$$\frac{1}{K} + \frac{j}{nK}$$

The corresponding impedance, looking right from 22, is

$$\frac{\frac{1}{K} - \frac{j}{nK}}{\left(\frac{1}{K}\right)^2 + \left(\frac{1}{nK}\right)^2}$$

Now, since nK, at a low frequency, is big compared with K, the term $\left(\frac{1}{nK}\right)^2$ in the denominator can be crossed out without introducing appreciable error. The expression for the impedance, looking right from terminals 22 then becomes

$$K - j\frac{K}{n}$$

But, at the frequency $\frac{f_0}{n}$ which is under consideration, the impedance of the inductance in the series arm is $+ j\frac{K}{n}$. The effect of this is to nullify the reactive component in the impedance seen from 22, so that the impedance, looking towards the right from 33, is non-reactive and equal to K. This result holds for all frequencies remote from cut off in the transmitting range and the network, used as a link between pieces of apparatus of resistance K, will transmit power without transmission loss at such frequencies.

At frequencies nearer the cut off frequency at which it is not permissible to neglect the term $\left(\frac{1}{nK}\right)^2$ in the expression for the impedance seen from 22, the impedance at 33 will only be non-reactive when the resistance closing terminals 11 is made greater than K—when the resistance is made equal to the mid-shunt characteristic impedance of the filter at the frequency under consideration. The impedance measured between terminals 33 under these conditions will be the corresponding mid-series characteristic impedance which is smaller than the nominal impedance K. For zero transmission loss the resistances of apparatus connected to 11 and 33 must be equal to the mid-shunt and mid-series characteristic impedances, respectively. As the cut off frequency is approached, to meet the conditions for zero transmission loss it is necessary still further to increase the one resistance and reduce the other until, finally, at the cut off frequency, the scheme breaks down.

In Fig. 8, attenuation constants have been plotted in decibels, even though in the resistanceless case, as has already been pointed out, attenuation of power cannot take place. The words "attenuation units" might perhaps have been used on the diagram in an attempt to allay scruples engendered by thoughts of the official definition of the decibel in terms of a power ratio. One would, however, be justified in pleading the resistanceless filter as a limiting case, for where even a little resistance is present the use of a unit so defined can be justified.

Use of Ratios to represent Impedances and Frequencies.

Curves of characteristic impedances and propagation constants have been drawn for a filter whose nominal impedance is 1000 ohms and whose cut off frequency is 1000 p.p.s. The corresponding equations have already been given in a form dependent upon the ratio $\frac{\omega}{\omega_0}$ and $\omega_0 L$, the nominal impedance. If, instead of plotting the curves against a frequency base, they are plotted against the ratio $\frac{\omega}{\omega_0}$, and impedances are plotted in terms of the nominal impedance K, the curves become of general application to all simple low-pass filters whatever their impedance and cut off frequency.

Input Impedance.

The input impedance Z_1 of a filter terminated at its output terminals with a resistance R is given by this formula

$$Z_1 = Z_{0A} \frac{1 - \dfrac{1 - \dfrac{R}{Z_{0B}}}{1 + \dfrac{R}{Z_{0B}}} \cdot e^{-2\Gamma}}{1 + \dfrac{1 - \dfrac{R}{Z_{0B}}}{1 + \dfrac{R}{Z_{0B}}} \cdot e^{-2\Gamma}} \quad \ldots\ldots (9)^*$$

where Γ is the propagation constant of the whole filter and Z_{0A} and Z_{0B} are the appropriate characteristic impedances at the sending end A and receiving end B, respectively. This formula assumes that the filter is terminated at ends A and B at mid-shunt or mid-series, where the ideal impedances are in the nature of resistances. One end can be terminated at mid-shunt and the other at mid-series, but in this case there will be an odd half-section for which the propagation constant is to be reckoned at half that for a complete section. The propagation constant of the whole filter is obtained by adding the propagation constants of the constituent sections and half-sections.

Input Impedance in Transmitting Range.

In the transmitting range of the filter (in the ideal case) the vector $e^{-2\Gamma}$ is of unit length but of changing angle. At frequencies at which $e^{\Gamma} = 0, 180^\circ, 360^\circ$, etc., the formula reduces to

$$Z_1 = Z_{0A} \times \frac{R}{Z_{0B}} \quad \ldots\ldots (10)$$

At frequencies where $e^{-\Gamma} = 90^\circ, 270^\circ$ . . . . . the formula becomes

$$Z_1 = Z_{0A} \times \frac{Z_{0B}}{R} \quad \ldots\ldots (11)$$

The input impedance is non-reactive at both sets of frequencies.

Taking the case where $Z_{0A} = Z_{0B}$ (*i.e.*, where both ends are terminated at mid-series or both at mid-shunt), at the first series of frequencies $Z_1 = R$ and at each of the second series $Z_1 = \dfrac{{Z_{0A}}^2}{R}$. These are points of maximum divergence on either side of the curve of characteristic impedance Z_{0A}.

* See Bibliography 3.

Fig. 10 shows the phase shift per section throughout the transmitting range of a simple low-pass filter. Below is the curve of mid-series characteristic impedance (Z_{0A}) together with the curve of $\frac{Z_{0A}^2}{R}$ for the case in which R = the nominal impedance (K) of the filter.

By reference to the upper curve, points can be marked out, corresponding to total phase shifts of 0°, 90°, 180°,

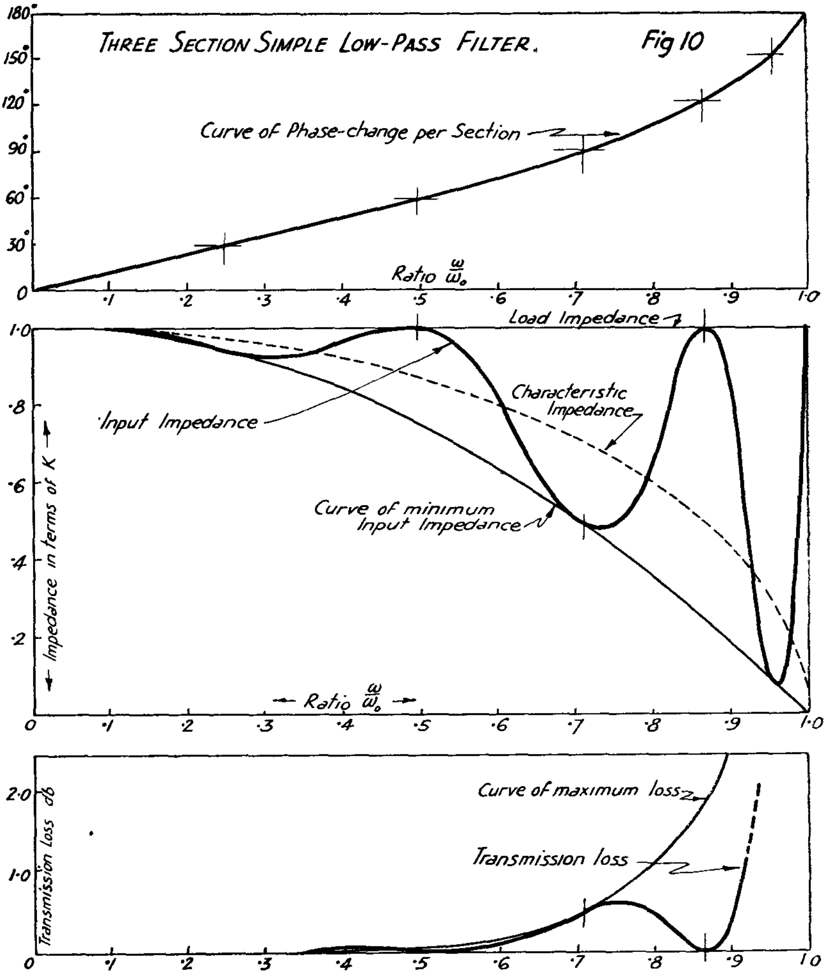


FIG. 10.

270° . . . . 540°, through which the curve of input impedance of the (three section) filter must pass. Drawing a fair curve through these points, oscillating between, and tangential to, K and the curve of $\frac{Z_{0A}^2}{R}$, an approximation to the curve of input impedance can be obtained.

Transmission Loss in the Transmitting Range.

The transmission loss, over the transmitting range, caused by the insertion of the filter between a generator of internal resistance R and the load resistance R, will be dependent in the ideal case on the impedance presented to the generator, since no power is lost in the filter. The loss corresponding to the curve $\frac{Z_{0A}^2}{R}$ has been calculated from the formula for terminal loss :—

$$T = 20 \log \left| \frac{R + Z}{2\sqrt{RZ}} \right| \text{db} \quad \text{.....(12)}$$

The transmission loss due to the filter will oscillate between this curve and zero, since when $Z_1 = R$ there will be no loss due to the interposition of the filter between the generator and the load resistance R.

The curve sketched will be an approximation to the theoretical transmission loss. It is seen that the loss does not exceed .6 db. up to a frequency equal to 90 per cent. of the cut off frequency. Above this point losses due to the resistance, which must be present in an actual filter, result in marked restriction of the amplitudes of the oscillations, but, by adding the loss corresponding to the total attenuation constant of the filter, a fair approximation to the curve of transmission loss of the filter would be obtained up to $.9\ \omega_0$.

Transmission Loss in the Attenuating Range.

In calculating the transmission loss in the attenuating range, if there are several sections, the effect of the load resistance R on the input impedance can be neglected since the filter is equivalent to a long line—long, that is, in the electrical sense. The formula given for calculating the terminal loss—equation (12)—can be used in this case, but it is necessary to remember that the quantities concerned are to be treated as vectors. Allowance must be made for terminal losses at each end of the filter. Curves of terminal loss are given in Fig. 11.

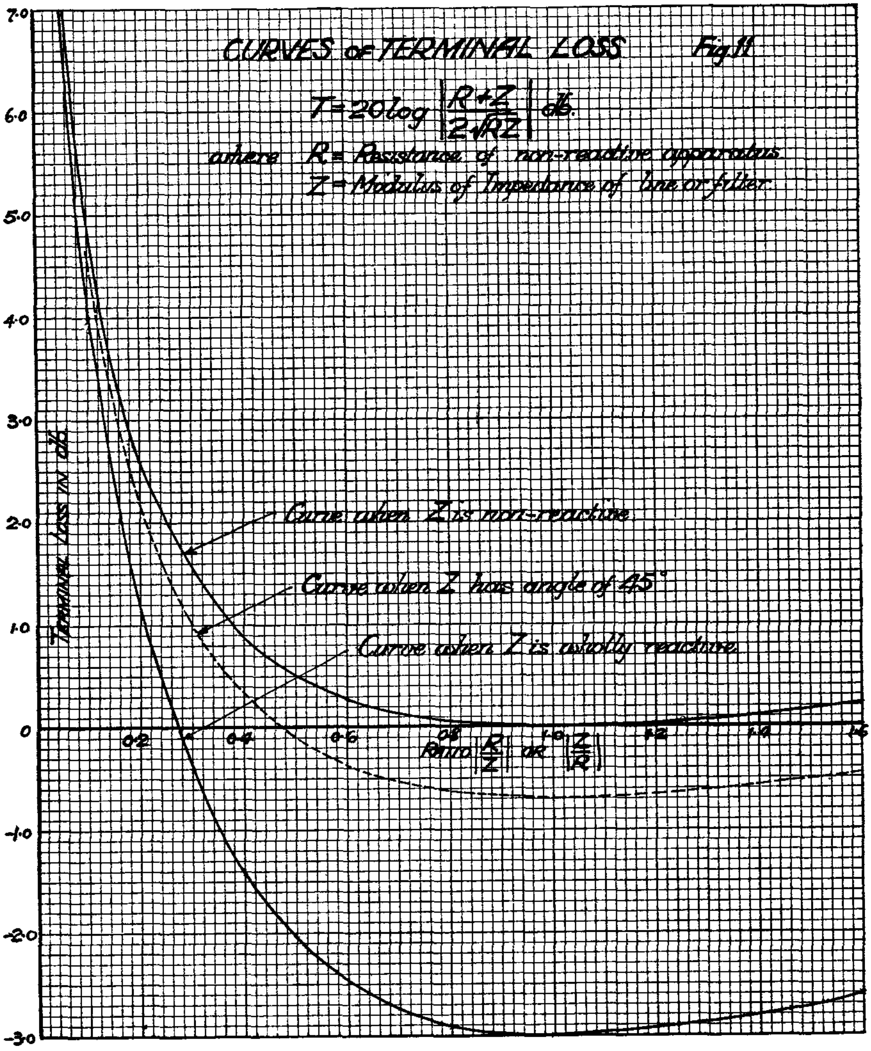


Fig. 11.

Fig. 12 shows the loss in the attenuating range of the three-section low-pass filter. The broken line corresponds to the total attenuation constant, converted to decibels, and it will be seen that the two curves differ by about 6 decibels.

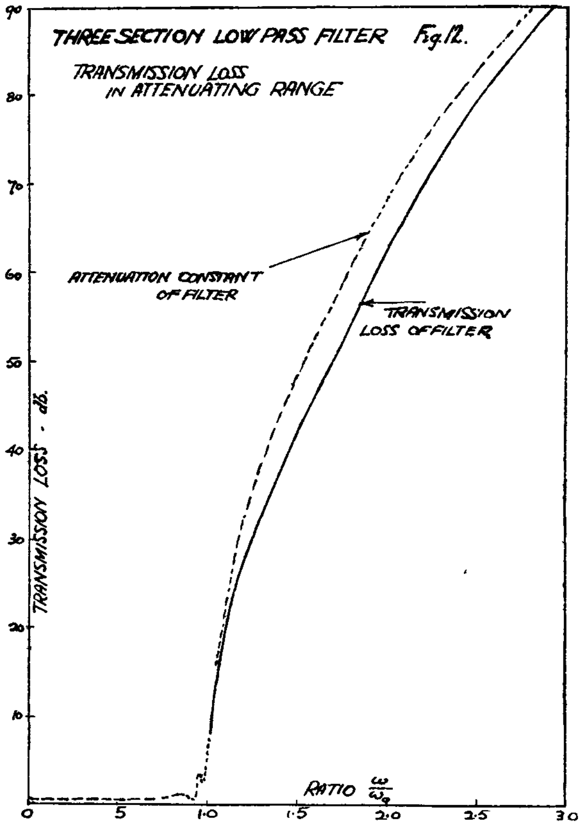


FIG. 12.

In general, the loss caused by a filter in the attenuating range, due to negative terminal losses, is less than what one might, speaking loosely, describe as the theoretical loss. As this difference is largely independent of the number of sections it will be realised that when, it may be, only one section is employed, the difference represents an important factor in the efficiency of suppression.

Simple High-Pass Filters.

The basic type of high-pass filter corresponding to the simple type of low-pass filter with which this paper has been dealing can be obtained by transposing the positions of inductances and condensers, as will be clear from a comparison of Figs. 6 and 7. If the elements of each half-section of a low-pass filter are transposed, the resulting high-pass filter will have the same cut off frequency and the same nominal impedance as the low-pass filter from which it has been formed. The attenuation and phase-shift curves of a simple high-pass filter are drawn in Fig. 9. As its name indicates, a high-pass filter passes freely all currents of higher frequency than the cut off while attenuating those of lower frequency. The curves corresponding to a ratio of resistance to reactance in the inductances of $\frac{1}{20}$, where they differ appreciably from the ideal curve, are shown by broken lines.

Nominal Impedance.

The nominal impedance of a high-pass filter is the characteristic impedance at frequencies much higher than the cut off frequency.

High-Pass and Low-Pass Filters Complementary.

It will be realised from a comparison of Figs. 6 and 7, and 8 and 9, that there is a strong family likeness between the high-pass and low-pass filters. The two filters are, in fact, complementary and the formulæ given for the low-pass filter will apply to the high-pass filter if altered by writing $\left(\frac{\omega_0}{\omega}\right)$, instead of $\left(\frac{\omega}{\omega_0}\right)$, wherever it occurs. The formula for the cut off frequency is, as already indicated, the same for the two filters.

After having dealt at some length with the simple low-pass filter, one feels that some apology may be necessary for dismissing the high-pass filter in so few words; but, in fact, the relationship between the two classes of filter is so close that no difficulty should arise in applying the discussion of the low-pass filter to the high-pass case. With the analogy of the loaded line in mind it is likely that a majority of electrical engineers are able to think more easily in terms of the low-pass filter than of the high-pass filter and accordingly the low-pass filter will be further considered.

Impedance at a point between Mid-Series and Mid-Shunt.

Reference has already been made to the alternation between the mid-series and mid-shunt characteristic impedances at successive junctions between half-sections. Fig. 13(*a*) represents a filter closed at each end by networks simulating the appropriate (mid-shunt) characteristic impedance. At 11, 22, 33, the filter has been shown with terminals at points of mid-section. At 44, however, the terminals have been placed at a position which is beyond the mid-shunt point. The impedance in the transmitting range, looking left from 44, is shown in Fig. 13 (*d*), the curve A being the real component and B representing the imaginary component. Looking right, the real component of the impedance is also represented by the curve A—the imaginary component is repre-

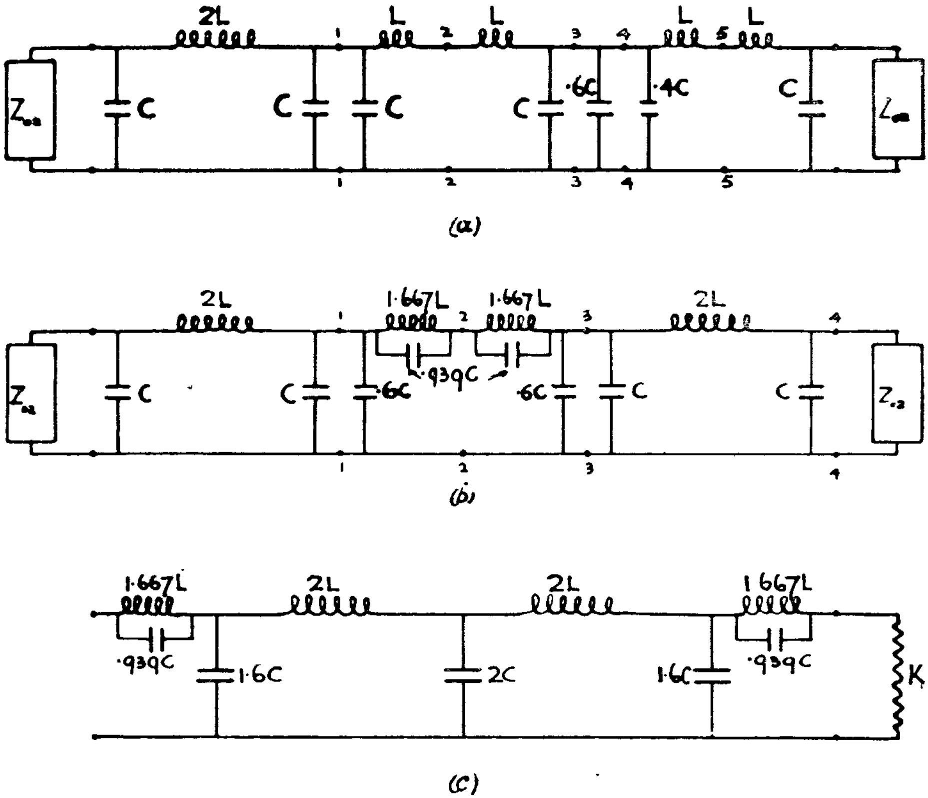


FIG. 13 (*a*), (*b*), (*c*).

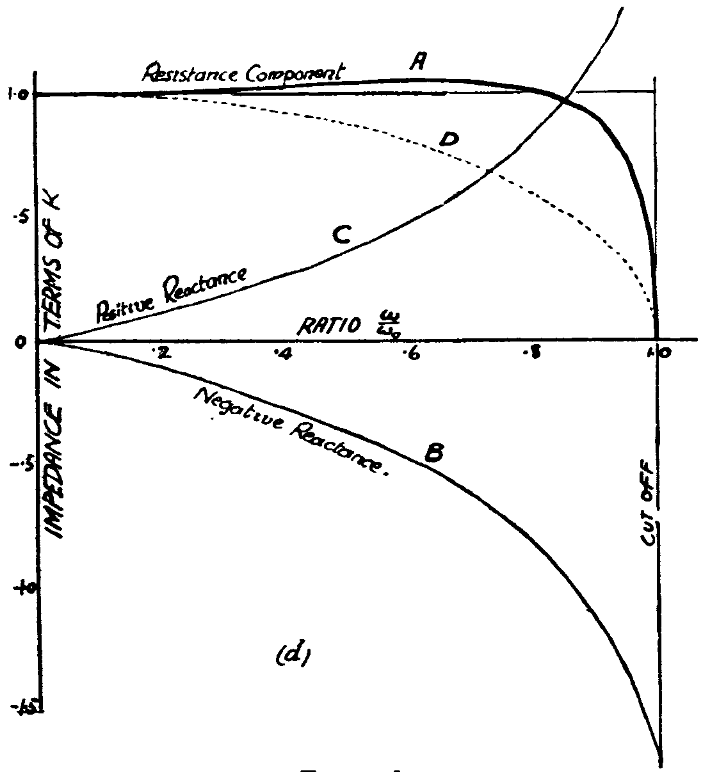


FIG. 13*d*.

sented by curve C which is a reflection of curve B. In short, the impedances in the two directions are conjugate. This relation exists in any ladder network for any lines of section, within the transmitting range. The diagram is shown, however, to draw attention to the shape of the A component curve. A filter with a non-reactive characteristic impedance curve of that shape would match up well with apparatus of constant resistance, and terminal reflections would be negligible, since the curve does not vary by more than 10 per cent. from K up to a frequency 10 per cent. below the cut off frequency. The addition of an inductance in series with the impedance, looking left, (curves A and B), would reduce the imaginary component considerably; and a carefully chosen parallel combination of inductance and capacity might be expected to achieve a much better result. Actually, a correctly proportioned parallel resonant circuit, placed in series with it, renders the impedance exactly non-reactive over the whole transmitting range.

Derived Section.

Using values of capacity and inductance, indicated in Fig. 13(*b*), between 11 and 22, a derived half-section is obtained which has a characteristic impedance at mid-series, *i.e.*, at 22, represented by the curve A in Fig. 13(*d*). (The normal mid-series impedance is shown by the broken curve D, for comparison).

Characteristic Impedances of Derived Section.

At mid-shunt, however, *i.e.*, at 11 and at 33, the characteristic impedance of the section is identical with the mid-shunt impedance of the ordinary simple low-pass filter, not only in the transmitting ranges but at all frequencies. Two half-sections placed to form a complete derived section are shown between two mid-shunt sections of the simple type. The impedances, looking right and left at 11 and also at 33, will therefore be equal.

Derived Section included in Filter.

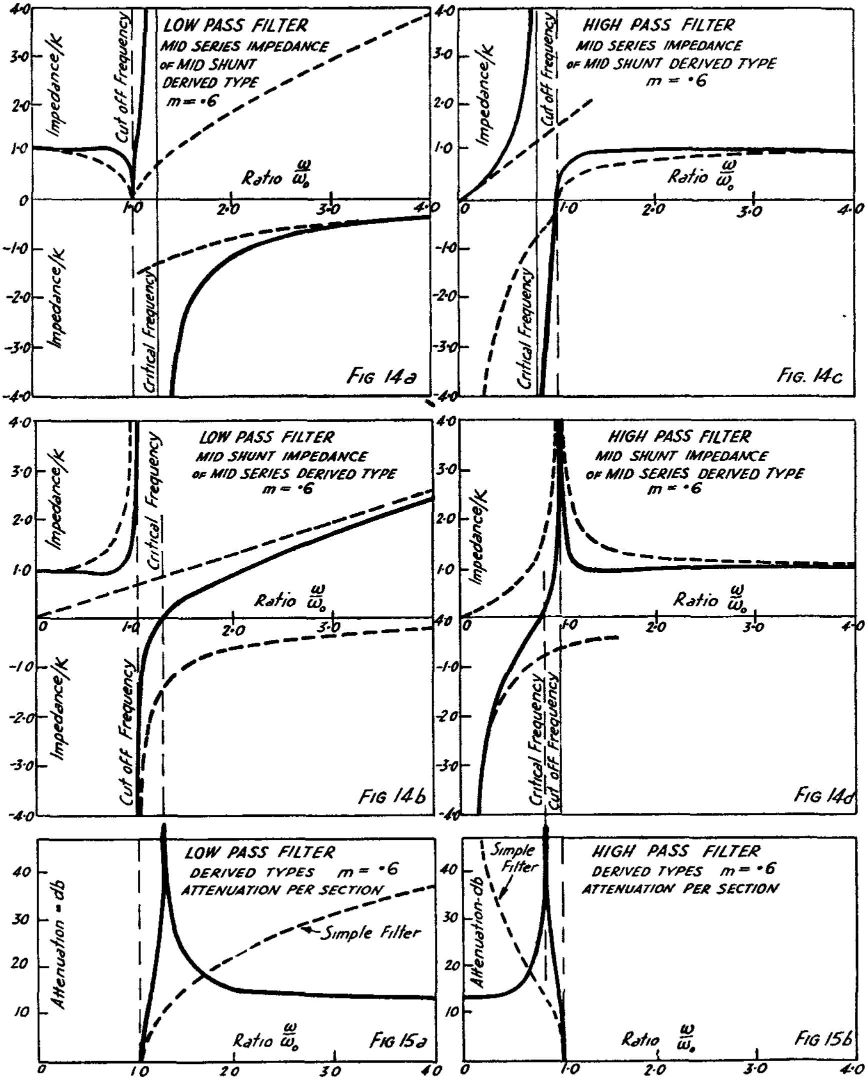
The derived section is included in the filter without giving rise to internal reflections at its junctions with the sections of simple type when current is transmitted through the filter.

Derived Section used to terminate a Filter.

The special character of the impedance of the derived section at mid-series can be used to terminate a filter designed to connect apparatus of constant resistance, giving but a negligible amount of reflection. This is illustrated in Fig. 13(*c*). The striking impedance properties of this type of section are accompanied by propagation characteristics which are shown in Fig. 15(*a*).

Propagation Characteristic.

The propagation characteristics of the simple low-pass filter are shown by broken lines for comparison. The attenuation constant of the derived section becomes infinite at the resonant frequency of the parallel circuit forming the series arm—infinite, one should say, in the purely theoretical case in which resistance is supposed to be absent. The maximum attenuation shown corresponds to that attainable with inductance coils in which the ratio $\frac{r}{\omega L} = \frac{1}{100}$. The impedance characteristic at mid-series is shown in Fig. 14(*a*). The theoretically infinite impedance which occurs at the point of resonance of the series arm, is accompanied by a change in sign.



FIGS. 14 & 15.

Attenuation after Critical Frequency.

The attenuation constant of the section from this point of resonance onwards is dependent upon the numerical ratio between the impedances constituting the series and shunt arms, which are henceforth of the same sign, as in a section of non-reactive artificial cable.

Mid-Shunt and Mid-Series Derived Sections.

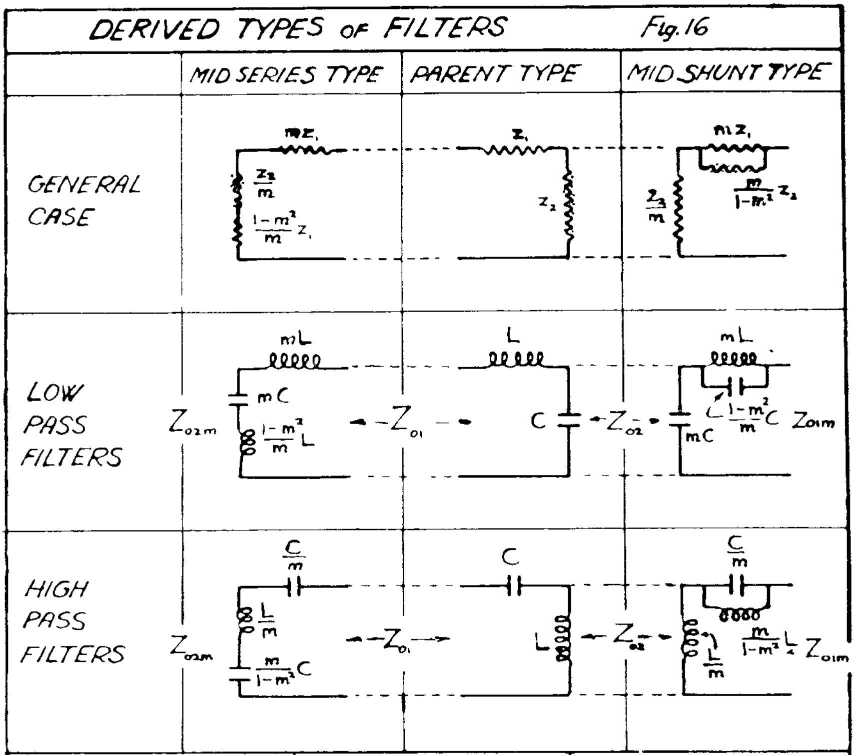
The derivation of special sections, such as the one discussed, is dealt with in Appendix 1. The type of section that has been described is known as the mid-shunt type of derived section because of its property of matching the impedance of the parent section at mid-shunt. The relations between the elements composing the parent section and the derived sections are illustrated in Fig. 16, together with the corresponding mid-series type of section, for high-pass, as well as low-pass, filters. The constant m can have any value between 0 and 1—for the simple parent types of filter, $m = 1$. The curves of impedance of derived filter types given in Fig. 14 will make clear the relations between the derived types. The propagation characteristics of the mid-series and mid-shunt types of sections having the same value of m are identical.

Attenuation Characteristics of Derived Sections.

The attenuation characteristics for a series of values of m are given in Fig. 20. The attenuation at the critical frequency in each case has been indicated to correspond with a ratio $\frac{r}{\omega L}$ in the inductances of .01 – the broken lines correspond to maximum attenuations for " dissipation constants " of .02 and .04.

Use of Derived Sections to obtain a desired Loss-Frequency Characteristic.

The special attenuation characteristics of derived sections can be employed to obtain almost any type of desired loss-frequency curve in the attenuating range by suitable combinations of these sections with the simple type of sections to form a filter. Thus, for instance, if it is desired to obtain a high attenuation at a frequency as near to the cut off as possible sections in which m has a low value, say, .4, should be included. By making use also of sections of simple type ($m = 1$) or sections in which m has a value of, say, .8, the



	LOW PASS FILTERS	HIGH PASS FILTERS
Cut Off Frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
Normal Impedance	$K = \omega_0 L = \frac{1}{\omega_0 C}$	$K = \omega_0 L = \frac{1}{\omega_0 C}$
Attenuation per Section	$\cosh \gamma = 1 - \frac{2m^2 \left(\frac{\omega}{\omega_0}\right)^2}{1-(1-m^2)\left(\frac{\omega}{\omega_0}\right)^2}$	$\cosh \gamma = 1 - \frac{2m^2 \left(\frac{\omega_0}{\omega}\right)^2}{1-(1-m^2)\left(\frac{\omega_0}{\omega}\right)^2}$
Frequency of Maximum Attenuation.	$\frac{\omega_1}{\omega_0} = \frac{1}{\sqrt{1-m^2}}$	$\frac{\omega_1}{\omega_0} = \sqrt{1-m^2}$
Normal Mid Series Imped'ce	$Z_{01} = K\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2}$	$Z_{01} = K\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2}$
Normal Mid Shunt Impedance	$Z_{02} = \frac{K}{\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2}}$	$Z_{02} = \frac{K}{\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2}}$
Mid Shunt Impedance of Mid Series Derived Type	$Z_{02m} = \frac{1-(1-m^2)\left(\frac{\omega}{\omega_0}\right)^2}{\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2}}$	$Z_{02m} = \frac{1-(1-m^2)\left(\frac{\omega_0}{\omega}\right)^2}{\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2}}$
Mid Series Impedance of Mid Shunt Derived Type	$Z_{01m} = \frac{\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2}}{1-(1-m^2)\left(\frac{\omega}{\omega_0}\right)^2}$	$Z_{01m} = \frac{\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2}}{1-(1-m^2)\left(\frac{\omega_0}{\omega}\right)^2}$

FIG. 16.

attenuation can be kept up to a predetermined value at frequencies remote from cut off. Such a composite filter can be considered as being formed from a uniform filter, composed of sections of the simple type, by replacing complete sections by the corresponding types of derived section. Thus a complete *mid-shunt* section of simple type can be replaced by a complete derived section of *mid-shunt type* which should have the same cut off frequency (ω_0) and the same nominal impedance (K) as the filter in which it is to be incorporated. If a heavy attenuation is required at a particular frequency, a section whose maximum attenuation occurs at this frequency would be included. The use of half-sections, in which m = .6, for the termination of filters has been mentioned.

Reciprocal Relation between Mid-Series and Mid-Shunt Derived Sections.

Fig. 17 gives curves of mid-series impedance of mid-

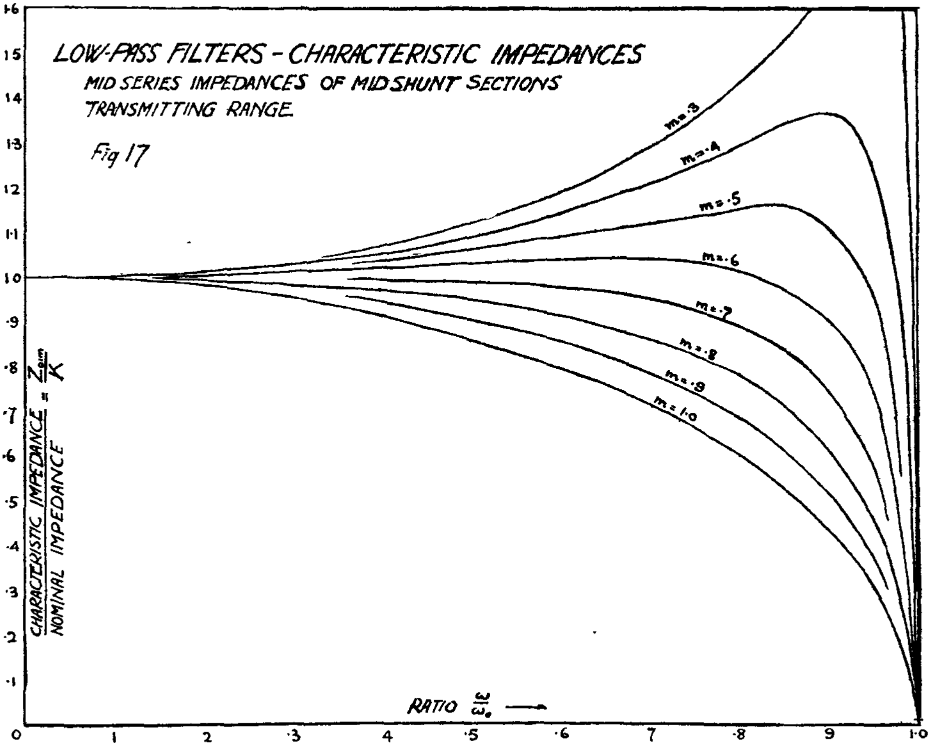


FIG. 17.

shunt types of derived section in the transmitting range. The corresponding curves for the mid-shunt impedance of the mid-series type of section will be reciprocal to this series of curves. Fig. 18 gives curves of mid-shunt impedance of mid-series derived sections in the attenuating range—the reciprocal relation holds also in this range.

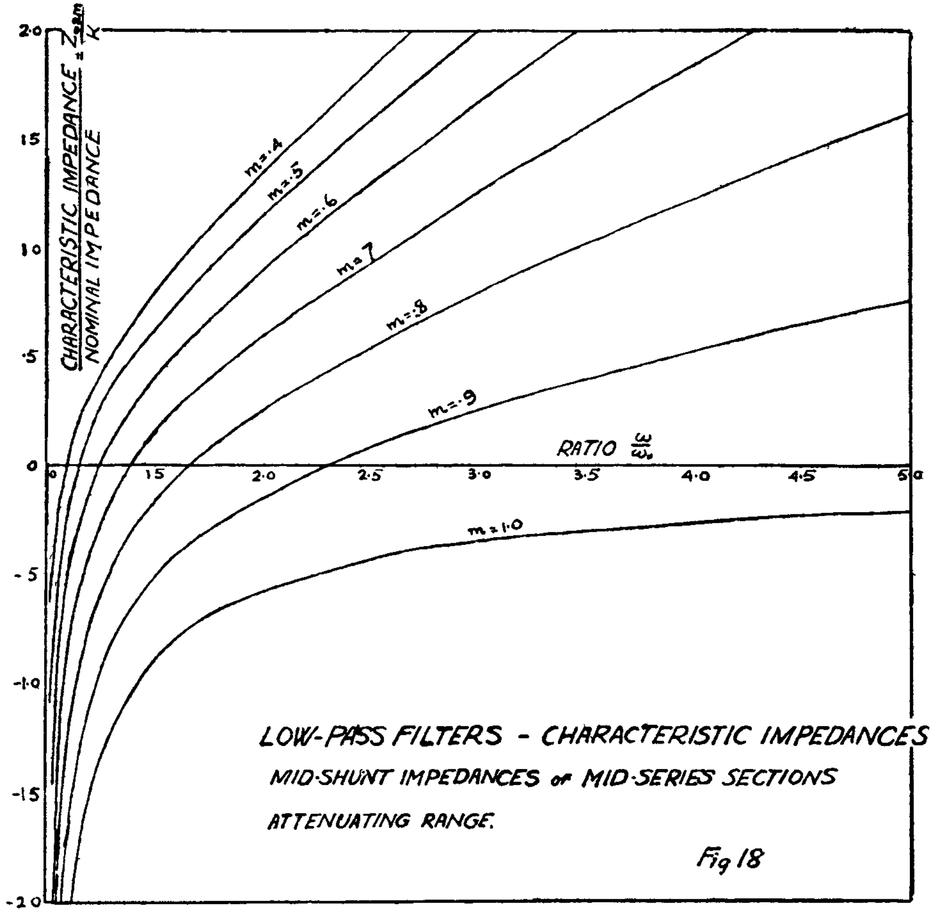
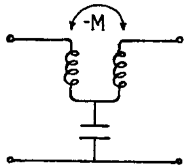


FIG. 18.

Mutual Inductance Filters.

Complete sections equivalent to the mid-shunt derived type can be formed by the use of negative mutual inductance by the circuit shown below :—



(g)

This form was used by the P.O. Engineering Department in early types of two-wire telephone repeaters. An interesting point is that, if positive mutual inductance be employed, the section produced corresponds to a value of m greater than 1. Some of these curves are shown in Figs. 19

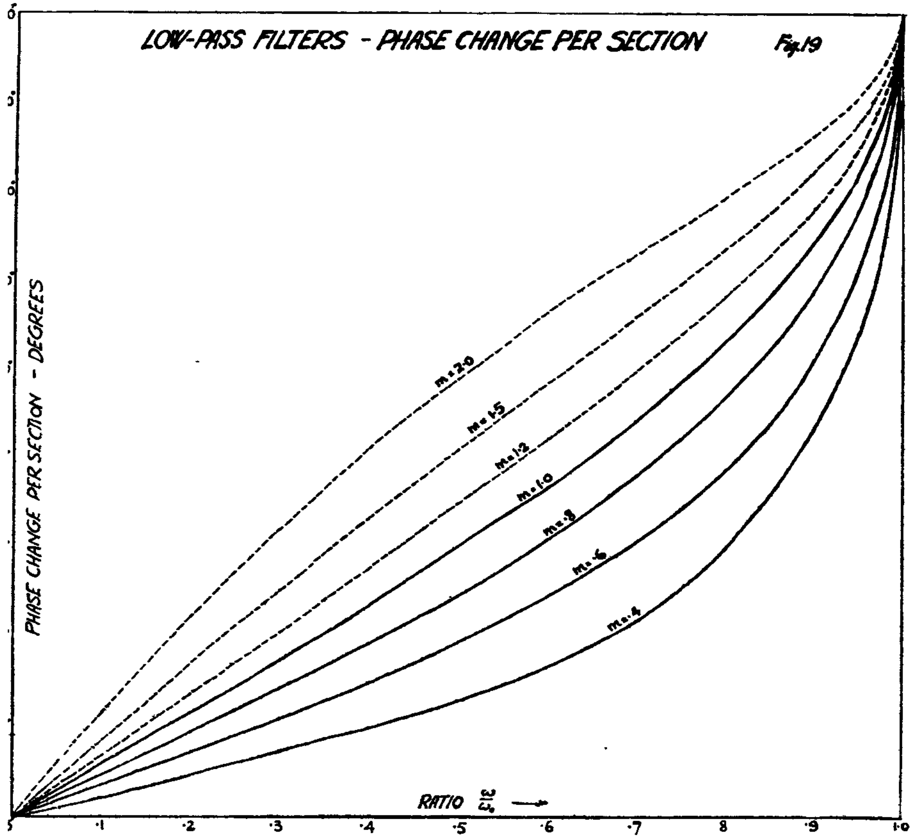


FIG. 19.

and 20. A method of calculating the values of the elements required to form such a section is given in Appendix II.

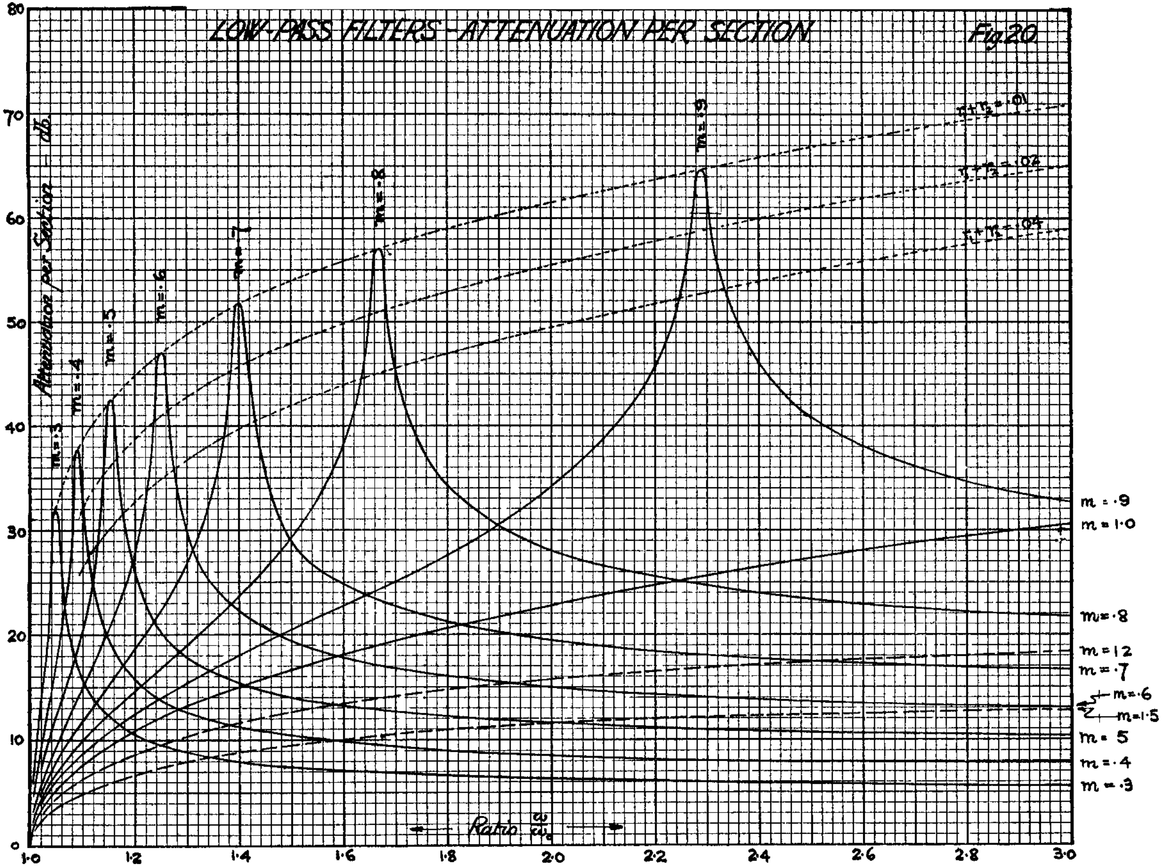


FIG. 20.

Relation between High-Pass and Low-Pass Filters.

As in the case of the parent types the equations for the corresponding types of high-pass filter can be obtained from the equations for low-pass derived filters by substituting $\left(\frac{\omega_0}{\omega}\right)$ for $\left(\frac{\omega}{\omega_0}\right)$ wherever this ratio occurs. Formulæ for low-pass and high-pass derived filters are shown together in Fig. 16. The curves for the low-pass "m-type" can be used for the corresponding high-pass types by reading $\left(\frac{\omega_0}{\omega}\right)$ for $\left(\frac{\omega}{\omega_0}\right)$ as the quantity represented by the abscissæ.

Use of Filters in Combination.

A filter can be employed to cut out unwanted frequencies or to reduce the efficiency of transmission of a range of frequencies, as, for instance, in a two-wire telephone repeater, as mentioned at the commencement of this paper; but probably the most important use to which filters are put is the subdivision of a band of frequencies available for transmission of signals into two or more separate channels, in such a way that mutual interference between the channels is kept within the limits of allowable cross-talk. This function is performed by filters in the cases of "Sub-Audio Telegraphy," "Voice-Frequency Telegraphy" and "Carrier-Current Telephony," to which cases attention has already been drawn.

In order that this shall be accomplished without appreciable losses due to reflections, it is necessary to arrange that the filters employed shall be terminated by impedances corresponding fairly well with their characteristic impedances throughout their transmitting ranges.

The usual method of separating two groups of frequencies and their direction into separate channels is indicated in Fig. 21 in the two lower diagrams.

High-Pass and Low-Pass Filters used in combination.

The two filters, high-pass and low-pass, usually have the same cut off frequency—all frequencies above f_0 pass from the line through the high-pass filter to apparatus of resistance K with which it is terminated. Frequencies below f_0 pass through the low-pass filter. The ends of the filters which are connected to the line are terminated at a point in the series arm which is beyond mid-series. The last series element in each of the filters normally has an impedance 1.6 times that of the series element of a true mid-series termination.

(The filters shown in Fig. 21 have been designed with the cut off frequency of the high-pass filter above that of the low-pass filter, and in this case a series element 1.5 times the normal value is more appropriate, but the principle remains the same).

In the transmitting range of the low-pass filter the extra inductance of ·6 L in the series arm, together with the input impedance of the high-pass filter, forms what is practically a half mid-shunt derived section, in which $m = \cdot 6$. (The high-pass filter, being in its attenuating range, has an input impedance roughly equivalent to the series capacity and the first shunt inductance in series). It has been pointed out earlier what an excellent termination to a filter can be formed by using a half-section, in which $m = \cdot 6$. The low-pass filter enters into the termination of the high-pass filter in a similar way, with the result that the joint impedance of the filters, seen from the line, is practically non-reactive over the whole frequency range and equal to K, except in the immediate neighbourhood of the cut off frequency. Both the high-pass and the low-pass filters are well matched as regards impedance throughout the transmitting range and transmission between the filters and the line takes place with a negligible amount of reflection. This arrangement of filters is used in carrier-current apparatus for the separation of speech currents and carrier-currents and also for separating higher frequency and lower frequency carrier-currents where this is necessary.

A combination of high-pass and low-pass filters having been used in this way to provide higher and lower channels, each of these channels can be further subdivided in the same way, but band-pass filters are generally used when a division into more than two channels is required.

No attempt has been made to deal with band-pass filters, but it should be mentioned that by combining a low-pass and a high-pass filter in series the effect of a single band-pass filter may be obtained. If this is done it is desirable to make use of m-type terminations to avoid reflections at the junctions of the two filters.

Example of Filter Design.

Fig. 21 has been included as an example of filter design—

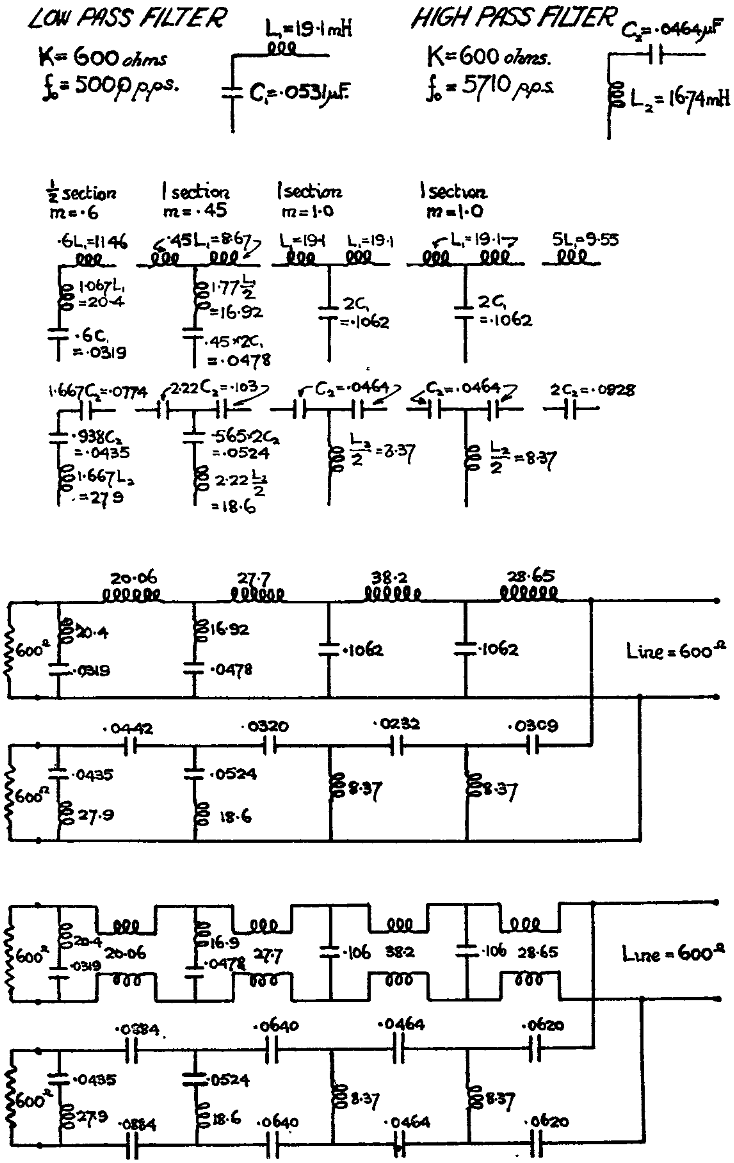

LOW PASS FILTER
K= 600 ohms
f₀= 5000 p.p.s.
L₁=19·1 mH
C₁=·0531μF
HIGH PASS FILTER
K=600 ohms.
f₀= 5710 p.p.s.
C₂=·0464μF
L₂=16·74mH
½ section m=·6
1 section m=·45
1 section m=1·0
1 section m=1·0
·6L₁=11·46
1·067L₁ =20·4
·6C₁ =·0319
·45L₁=8·67
1·77 L₁/2 =16·92
·45×2C₁ =·0478
L₁=19·1
L₁=19·1
2C₁ =·1062
L₁=19·1
2C₁ =·1062
5L₁=9·55
1·667C₂=·0774
·938C₂ =·0435
1·667L₂ =27·9
2·22C₂=·103
·565×2C₂ =·0524
2·22 L₂/2 =18·6
C₂=·0464
L₂/2 =8·37
C₂=·0464
L₂/2 =8·37
2C₂=·0928
20·06
27·7
38·2
28·65
600Ω
20·4
·0319
16·92
·0478
·1062
·1062
Line = 600Ω
·0442
·0320
·0232
·0309
600Ω
·0435
27·9
·0524
18·6
8·37
8·37
600Ω
20·4
·0319
20·06
16·9
·0478
27·7
·106
38·2
·106
28·65
Line = 600Ω
·0884
·0640
·0464
·0620
600Ω
·0435
27·9
·0884
·0524
18·6
·0640
8·37
·0464
8·37
·0620

FIG. 21.

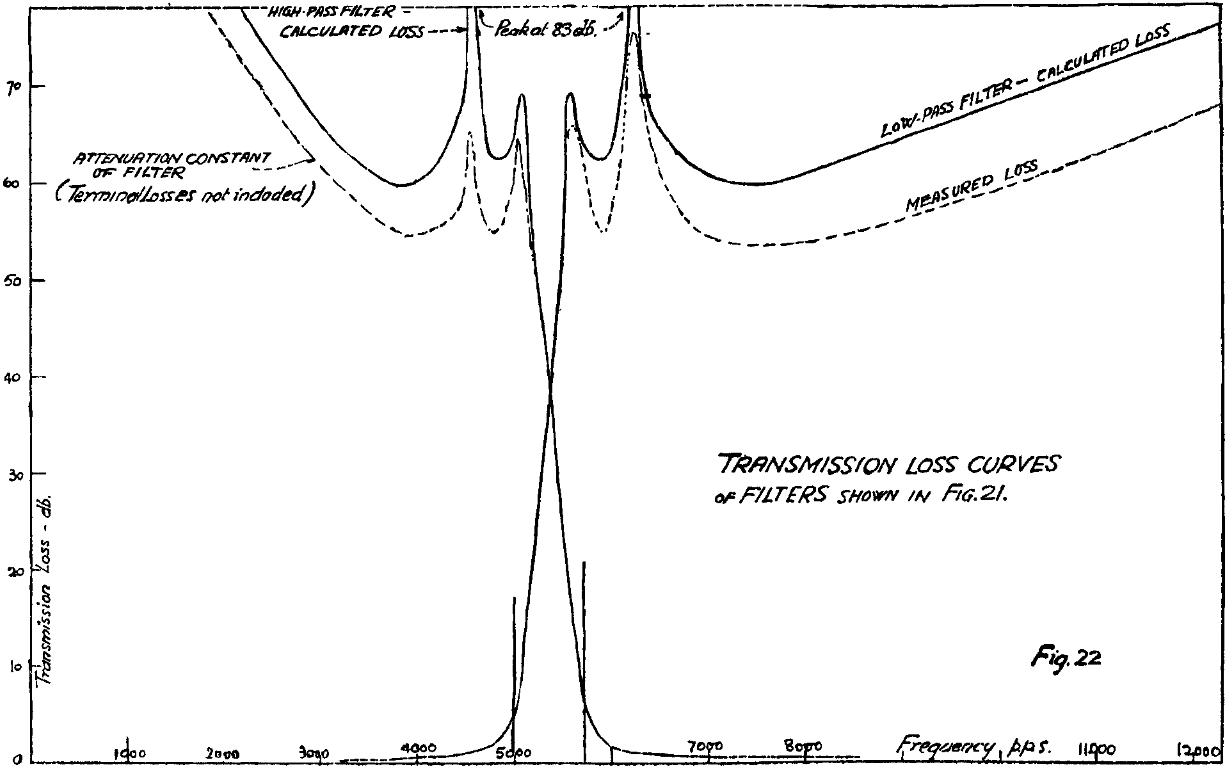

HIGH-PASS FILTER — CALCULATED LOSS
Peak at 83 db.
LOW-PASS FILTER — CALCULATED LOSS
ATTENUATION CONSTANT OF FILTER (Terminal losses not included)
MEASURED LOSS
TRANSMISSION LOSS CURVES of FILTERS SHOWN IN FIG. 21.
Fig. 22
Transmission Loss - db.
Frequency, p.p.s.
0
10
20
30
40
50
60
70
1000
2000
3000
4000
5000
7000
8000
11000
12000

FIG. 22.

the development of the structure from the component sections and half-sections is shown and will afford an illustration of the building up of a composite filter.

The calculated performances of the two filters, taking account of terminal losses in the attenuating ranges, for transmission between the line and the terminal apparatus, are shown in Fig. 22 by the full lines. In the case of the low-pass filter a record of the measured transmission loss is shown by the broken line. The reason for the discrepancy in the attenuation curves is not known, but the agreement in the size of the peaks and the general shape of the curve is good. The ratio of $\frac{r}{\omega L}$ was taken at $\frac{1}{100}$, which is somewhat better than the ratio for the actual coils used in the filter, but this would only affect the curve by emphasising the height of the peaks. The broken line in the high-pass case shows the total attenuation constant of the filter. The net terminal loss with this combination of filters is positive in the attenuation range, due partly to the low resistance of the line in parallel with the real input impedance of the other filter which is in its transmitting range, and also to the fact that the current which does reach the end of the filter is divided between these two resistances. The terminal loss at the critical frequency of the terminal half-section adds 13 db. to the transmission loss at this frequency.

The matter that has been presented is, for the most part, available in books and periodicals, but it is hoped that this paper may be found useful as an introduction to the study of electric filters. It is, to some, a subject of absorbing interest, but it is difficult to communicate one's enthusiasm for what may appear to be, at first sight, complicated and dry abstractions.

In conclusion, it is fitting to pay tribute to the genius of Otto J. Zobel. The theory underlying the derivation of the " m-type " sections and their use to eliminate reflections and in the production of filters with almost any desired transmission characteristics, surely form a piece of work, which for beautiful simplicity, is equalled in the field of electrical communication only by Carson's system of carrier-current transmission.

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2. " Theory and Design of Uniform and Composite Electric Wave Filters." Otto J. Zobel. B.S.T.J., January, 1923.

3. " Transmission Characteristics of Electric Wave Filters." Otto J. Zobel. B.S.T.J., October, 1924.

   This article contains a valuable examination of the general problem of transmission through a line or a network such as a filter. The formula given in the present paper for the sending end impedance of a filter can be derived from the well known transmission formulæ, but Zobel's examination of the problem is made without recourse to hyperbolic functions. In the present paper the term " characteristic impedance " has been used in the sense for which the term " image impedance " is defined by Zobel—the section on characteristic impedance in Zobel's article has no bearing on problems touched upon in the present paper.

4. " Mutual Inductance in Wave Filters with an introduction on Filter Design." K. S. Johnson and T. E. Shea. B.S.T.J., January, 1925.

5. " Transmission Networks and Wave Filters." T. E. Shea, published by Chapman & Hall.

6. " Das Dampfungsmasz der Pupinleitung." H. F. Mayer. T.F.T., June, 1927.

   Characteristic curves of loaded lines given in the present paper were calculated from formulæ in this article which are probably the most useful which have been developed for calculations of this kind. A method is given for the calculation of the propagation constant from cosh γ, expressed in the form $A + jB$.

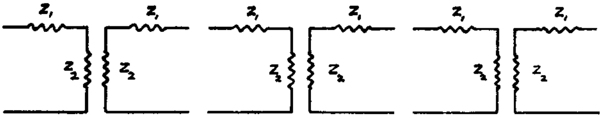
7.* " The Theory of Electrical Artificial Lines and Filters." A. C. Bartlett, B.A., published by Chapman & Hall.

8.* " Les Filtres Electriques." P. David, published by Gauthier-Villars, 1926.

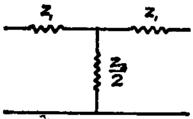
* Added at the suggestion of Mr. C. Robinson, E.-in-C.O.

APPENDIX I.

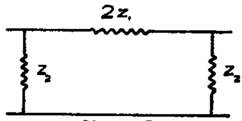
Characteristic Impedances and Propagation Constants of Ladder Type Networks.



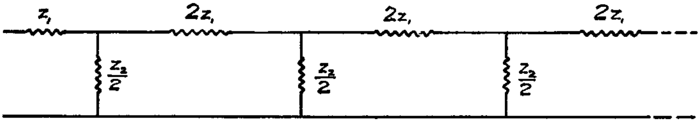
(a)



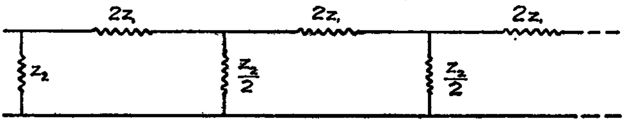
Mid Series Section
(b)



Mid Shunt Section
(c)



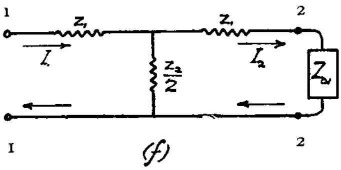
(d)



(e)

Let the diagram (*a*) represent a series of " half-sections," the impedance of each series element being denoted by z_1, and that of the shunt elements by z_2. By combining two half-sections, symmetrical mid-series and mid-shunt sections can be formed as shown at (*b*) and (*c*) respectively. By combining the whole series of half-sections the recurrent structure of (*d*) is obtained. This can be imagined as built up of a succession of mid-series sections identical with the section shown at (*b*), but if the network is terminated at a mid-shunt point, as shown at (*e*) the network can be considered to be built up of a succession of mid-shunt sections similar to the section shown at (c).

If the network at (*d*) is imagined to consist of an infinitely large number of sections, the impedance measured between terminals 1.1. will be the mid-series characteristic impedance of the network, which in this paper is denoted by Z_{01}.



(*f*)

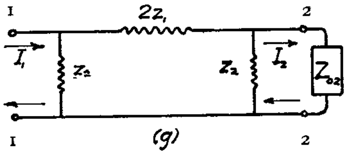
At (*f*) is shown a mid-series section with terminals 2.2. closed through its characteristic impedance Z_{01}. The impedance that would be measured between terminals 1.1. would be Z_{01}. (This arrangement is equivalent to adding one more section in front of the infinite series at (*d*)).

The impedance between 1.1. is equal to the impedance of the series arm (z_1) plus the impedance of z_1 and Z_{01} in shunt with $\frac{z_2}{2}$. The following equation may, then, be stated:—

$$Z_{01} = z_1 + \frac{\frac{z_2}{2}(z_1 + Z_{01})}{\frac{z_2}{2} + z_1 + Z_{01}}$$

From this the mid-series characteristic impedance is found to be:—

$$Z_{01} = \sqrt{z_1 z_2 + z_1^2} \quad \text{.......................................(1)}$$



(*g*)

The corresponding expression for the mid-shunt characteristic impedance can be obtained in a similar way. (*g*) represents a mid-shunt section terminated by its characteristic impedance Z_{02}. The impedance of z_2 in parallel with Z_{02} is $\frac{z_2 Z_{02}}{z_2 + Z_{02}}$. The addition of $2z_1$ in series brings the impedance to $\frac{2z_1 z_2 + Z_{02}(2z_1 + z_2)}{z_2 + Z_{02}}$

Putting z_2 in parallel with this impedance we may write:

$$Z_{02} = z_2 \frac{2z_1z_2 + Z_{02}(2z_1 + z_2)}{z_2^2 + 2z_1z_2 + 2Z_{02}(z_1 + z_2)}$$

and from this equation the expression for the mid-shunt characteristic impedance is found to be :—

$$Z_{02} = \frac{z_1z_2}{\sqrt{z_1z_2 + z_1^2}} \quad \text{.........(2)}$$

Considering the diagram at (*f*), if I_1 and I_2 are the currents at 1. and 2. respectively, when a source of power maintains a voltage between terminals 1.1., the propagation constant, γ, of the section may be defined by the equation :—

$$e^{\gamma} = \frac{I_1}{I_2} \quad \text{.........(3)}$$

An expression for e^{γ} can be obtained as follows :—

The current I_1 divides at mid-section between the shunt and series paths in inverse proportion to their impedances, $\frac{z_2}{2}$ and $z_1 + Z_{01}$ respectively) :—

$$\frac{I_1}{I_2} = \frac{\frac{z_2}{2} + z_1 + Z_{01}}{\frac{z_2}{2}} = 1 + \frac{2z_1}{z_2} + \frac{2Z_{01}}{z_2} \quad \text{.........(4)}$$

Substituting for Z_{01} from equation (1)

$$e^{\gamma} = \frac{I_1}{I_2} = 1 + \frac{2z_1}{z_2} + \frac{2\sqrt{z_1z_2 + z_1^2}}{z_2} \quad \text{.........(5)}$$

In the case of a filter such as a simple low-pass filter z_1 and z_2 are both imaginary and of opposite sign. (This is so in the case of any filter in the transmitting range.) Writing $j|z_1|$ for z_1 and $-j|z_2|$ for z_2 :—

$$\frac{I_1}{I_2} = 1 - 2\left|\frac{z_1}{z_2}\right| + \frac{2\sqrt{|z_1z_2| - |z_1^2|}}{-j|z_2|} \quad \text{.........(6)}$$

The expression $\sqrt{|z_1z_2| - |z_1^2|}$, (mid-series characteristic impedance), is real so long as $|z_2| > |z_1|$—The third term, however, due to the presence of j in the denominator, is unreal. The magnitude of $\frac{I_1}{I_2}$ is therefore given by the expression :—

$$\left|\frac{I_1}{I_2}\right| = \sqrt{\left[1 - 2\left|\frac{z_1}{z_2}\right|\right]^2 + \left[\frac{2\sqrt{|z_1z_2| - |z_1^2|}}{|z_2|}\right]^2}$$

$$= \sqrt{1 - 4\left|\frac{z_1}{z_2}\right| + 4\left|\left(\frac{z_1}{z_2}\right)^2\right| + \frac{4|z_1z_2| - 4|z_1^2|}{|z_2^2|}}$$

The above expression for $\frac{I_1}{I_2}$ reduces to unity; so that if $|z_2| > |z_1|$, (these impedances being of opposite sign), currents are not attenuated during their passage through the filter section. (It is to be remembered that the section under consideration is supposed to be terminated by its characteristic impedance).

When $z_2 < z_1$ all the terms on the right hand side of equation (6) are real and

$$\left|\frac{I_1}{I_2}\right| > 1$$

That is attenuation takes place, since the current leaving the section at 2.2. is less than that entering at 1.1.

In practice, it is convenient to calculate the value of cosh γ and thence to determine γ. The required formula can be found by making use of a well known relation which applies if the terminals 2.2., in diagrams (*f*) or (*g*), are short-circuited, namely :—

$$\frac{I_1}{I_2} = \cosh \gamma \; *$$

Considering the mid-shunt section of diagram (*g*) :— Terminals 2.2. are to be supposed short-circuited. The current I_1 will divide in inverse ratio to the impedances z_2 and $2z_1$ of the two parallel paths, whence :—

$$\frac{I_1}{I_2} = \frac{2z_1 + z_2}{z_2}$$

$$\text{or } \cosh \gamma = 1 + \frac{2z_1}{z_2} \quad \ldots\ldots\ldots\ldots(7)$$

Writing $j|z_1|$ for z_1 and $-j|z_2|$ for z_2 :—

$$\cosh \gamma = 1 - 2\left|\frac{z_1}{z_2}\right|$$

It can be shown, from consideration of the expanded expression for the hyperbolic cosine of a complex quantity, ($\cosh (a + jb)$) that when cosh γ is real and lies between the values $+1$ and -1 that $e^{\gamma} = 1$. That is, attenuation in the "infinite line" condition does not take place.* At frequencies at the boundary between the transmitting and attenuating ranges—"cut-off" frequencies—where cosh γ $= +1$ or -1, by equation (7) :—

$$\frac{z_1}{z_2} = 0 \text{ or } -1$$

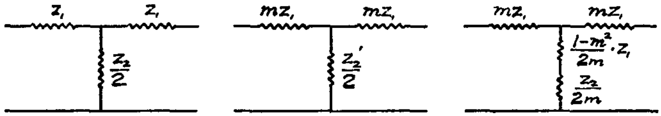
In obtaining values of γ in the attenuating range, for the ideal resistanceless case, from tables of hyperbolic functions it is to be noted that when cosh γ is negative and < -1, γ has

* See J. G. Hill's "Telephone Transmission," page 365.

* See page 16 for the interpretation to be assigned to γ within this range.

the same value as in the case of the corresponding positive value of cosh γ, but that the attenuation is accompanied by a phase-change of 180°.

Derived Sections.



(h) (k) (l)

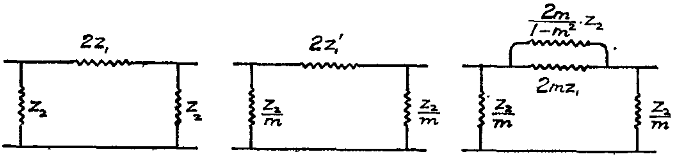
At (h) a normal mid-series filter section is depicted—at (k) a section of the same form is shown, but the series arms have impedances m times as great as those of the normal section. The shunt arm impedance has been called $\frac{z_2'}{2}$ If the expressions for the mid-series characteristic impedance of the two sections are equated a value for $\frac{z_2'}{2}$ can be found so that the sections at (a) and (b) shall have the same characteristic impedances. Then applying equation (1):—

$$\sqrt{z_1 z_2 + z_1^2} = \sqrt{m z_1 z_2' + m^2 z_1^2}$$

$$\text{Whence } z_2' = z_1 \cdot \frac{1 - m^2}{m} + \frac{z_2}{m} \quad \text{..................(8)}$$

The new section, having the same characteristic impedance as the "parent" section at (h) is shown at (l) and is known as a mid-series type derived section. The reduction in the impedance of the series arm, (for m is normally less than unity for a realisable network), requires, in addition to a corresponding increase in the value of the shunt impedance to $\frac{z_2}{2m}$, the addition of an element in series with it of the same nature as the series arm impedance Z_1.

A corresponding mid-shunt type of derived section can be found by a similar method.



(n) (p) (q)

The figure shows the parent mid-shunt section at (n) and a section at (p) in which the shunt elements have been altered to $\frac{z_2}{m}$. Equating the characteristic impedances obtained in accordance with equation (2):—

$$\frac{z_1 z_2}{\sqrt{z_1 z_2 + z_1^2}} = \frac{z_1' \frac{z_2}{m}}{\sqrt{z_1' \frac{z_2}{m} + z_1'^2}}$$

the following relation can be found:—

$$z_1' = \frac{1}{\frac{1}{m z_1} + \frac{1}{\frac{m z_2}{1 - m^2}}} \quad \text{.....................................(9)}$$

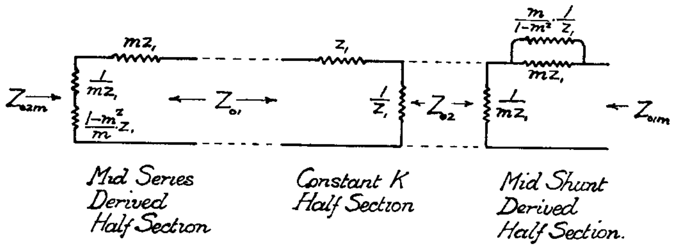
This equation will be found to be compatible with the derived section shown at (q). This section has the same characteristic impedance as the parent section at (n).

Applying these results to filter networks, since the derived sections have impedances which must correspond at all frequencies with those of the parent sections, the derived sections' impedances must change from real to imaginary at the same frequency as that of the parent section. The transmitting and attenuating ranges therefore correspond. As, however, equation (7) indicates that the propagation constant depends upon the ratio between the series and shunt impedances of a section, the derived sections will necessarily have different propagation characteristics from those of the parent type of section.

Simple low-pass and high-pass filters—filters in which each series and shunt element consists of either inductance or capacity—not a combination of these—have the property that the product of the impedances of the shunt and series elements is a constant at all frequencies. Thus in the case of a simple low-pass filter $z_1 z_2 = \frac{j\omega L}{j\omega C} = \frac{L}{C} = K^2$ where K is the "nominal impedance" of the filter. Such filters have been described as of "constant K" type by Zobel. There are "constant K" types of band-pass, band elimination and other types of filters and in general the constant K type in which series and shunt elements are "reciprocal networks" may advantageously be considered to form the basic type for filters having given frequency bands of transmission and suppression.

It will now, therefore, be assumed that the networks shown in diagrams (*a*) to (*e*) are of " constant K " type. Reference has been made in the main part of the paper to the more general application of impedance curves when plotted in terms of nominal impedance K. If in considering a particular filter, impedances are stated in terms of units of the magnitude of K, in the case of a constant K type we have the relation $z_1 z_2 = 1$. This, while in the author's opinion at least, simplifying the work of deriving formulæ, will introduce no difficulties if in applying the results so obtained it is remembered that formulæ denoting impedances require to be multiplied by K.

Making use, then, of the relation $z_1 z_2 = 1$ the diagram (*r*) has been drawn showing a half-section of constant K type linked correctly with half-sections of the mid-series and mid-shunt type.



(*r*)

Considering the mid-series derived section, writing:

mz_1 in place of z_1

and $\frac{1 - m^2}{m} z_1 + \frac{1}{mz_1}$ in place of Z_2 in equation (2) of this appendix the following equation results for the mid-shunt impedance of the mid-series derived section:—

$$Z_{02m} = \frac{1 + (1 - m^2) z_1^2}{\sqrt{1 + z_1^2}} \quad \text{.................................(10)}$$

Note that substituting $\frac{1}{z_1}$ for z_2 in equation (2) or putting $m = 1$ in equation (10) the expression for the mid-shunt characteristic impedance of the constant K (parent) section is obtained:

$$Z_{02} = \frac{1}{\sqrt{1 + z_1^2}} \qquad (11)$$

The relation between the mid-series impedance of the derived mid-shunt section and the mid-series impedance of the constant K network can be obtained in a similar fashion; these are:

Mid-series impedance of a mid-shunt derived section:—

$$Z_{01m} = \frac{\sqrt{1 + z_1^2}}{1 + (1 - m^2)z_1^2} \qquad (12)$$

Mid-series impedance of constant K section:—

$$Z_{01} = \sqrt{1 + z_1^2} \qquad (13)$$

The reciprocal relation between Z_{01m} and Z_{02m} is noteworthy as is also that between Z_{01} and Z_{02}.

The propagation constant for a full section of the mid-series and of the mid-shunt derived types is found to be:—

$$\cosh \gamma_m = 1 + \frac{2m^2 z_1^2}{1 + (1 - m^2)z_1^2} \qquad (14)$$

and the propagation constant for a section of the parent constant K type is given by

$$\cosh \gamma = 1 + 2z_1^2 \qquad (15)$$

As to the interpretation of these equations when applied to low-pass filters. It has been shown in the main part of the paper (equation 3) that $K = \omega_0 L$.

In equations (10) to (13) of this appendix impedances are all stated in terms of units having a magnitude = K.

In consequence, in these equations, z_1 signifies, in the case of a low-pass filter, not $j\omega L$, but $\frac{j\omega L}{K}$, that is:—

$$\left.\begin{aligned} z_1 &= \frac{j\omega L}{\omega_0 L} = j\frac{\omega}{\omega_0} \\ \text{and} \quad z_1^2 &= -\left(\frac{\omega}{\omega_0}\right)^2 \end{aligned}\right\} \qquad (16)$$

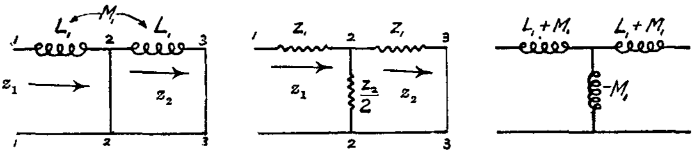
In the case of high-pass filters:—

$$z_1 = \frac{1}{j\omega C} \times \frac{1}{K} = \frac{\omega_0 C}{j\omega C} = -j\frac{\omega_0}{\omega}$$

$$\text{and} \quad z_1^2 = -\left(\frac{\omega_0}{\omega}\right)^2 \qquad (17)$$

APPENDIX II.

Low-Pass Filters with Mutual Inductance.



(a) (b) (c)

Let the diagram at (*a*) represent two equal coils of inductance L_1 with mutual inductance M_1 between them. It is desired to determine the impedances of the elements of the equivalent mid-series section represented at (*b*). Terminals 33 have been shown short-circuited in both diagrams. Let i_1, i_2, etc., represent the currents flowing. In mesh 2,3 of (*a*), equating the sum of the voltages around the mesh to zero :—

$$O = i_2 j\omega L_1 + i_1 j\omega M_1$$

whence $i_2 = - i_1 \times \dfrac{M_1}{L_1}$ and $i_1 - i_2 = i_1 \left(1 + \dfrac{M_1}{L_1} \right)$

Considering diagram (*b*), since the current i_1 divides between the branches 22 and 23 in inverse ratio to their impedances and

$$\frac{z_1}{\frac{z_2}{2}} = \frac{i_1 - i_2}{i_2} = - \frac{M_1 + L_1}{M_1}$$

$$\therefore \frac{z_2}{2} = - z_1 \times \frac{M_1}{M_1 + L_1} \qquad \text{(1)}$$

Now if terminals 33 are open circuited, equating the impedances between terminals 1.1. in diagrams (*b*) and (*a*)

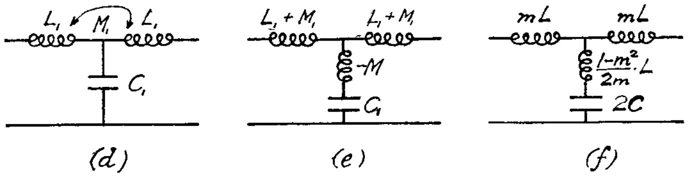
$$z_1 + \frac{z_2}{2} = j\omega L_1 \text{ and } \therefore \text{ by (1) } \frac{L_1}{L_1 + M_1} z_1 = j\omega L_1$$

$$\left. \begin{aligned} &\text{and } z_1 = j\omega(L_1 + M_1) \\ &\text{By (1) } \frac{z_2}{2} = - j\omega M_1 \end{aligned} \right\} \qquad \text{(2)}$$

The equivalent network is represented at (*c*).

If the general case, in which impedances are added in series and shunt arms of the network of diagram (*a*) is considered, the same values of equivalent inductance result.

The filter section shown at (*d*) is equivalent to the section shown at (*e*).



A mid-series derived section is shown at (*f*). In order that the sections (*d*) and (*f*) shall be equivalent :—

$$mL = L_1 + M_1$$

$$\frac{1 - m^2}{2m} L = - M_1 \qquad \text{.............(3)}$$

$$\left.\begin{aligned} L_1 &= \frac{1 + m^2}{2m} L \\ \text{and by (3)} \quad \frac{M_1}{L_1} &= - \frac{1 - m^2}{1 + m^2} \end{aligned}\right\} \qquad \text{.............(4)}$$

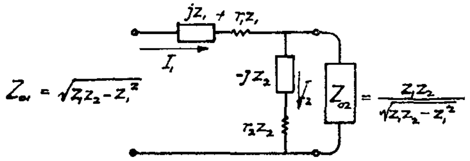
In the normal case when m is < 1 the mutual inductance is required to be negative. When the sign of the mutual inductance is positive m has values > 1 and the attenuation after the cut off frequency rises less rapidly than is the case with the simple low-pass filter. When there is full coupling between the series inductances the cut off frequency recedes to infinite frequency and the network becomes a phase-changing section.

APPENDIX III.

Attenuation of Filter Sections in the Transmitting Range.

Approximate expressions for the attenuation in the transmitting range of the simple and derived types of filter sections can be obtained by the following argument.

Consider first the general case of a half-section, the mid-shunt terminals being closed by the mid-shunt characteristic impedance. The series and shunt elements are, as before, denoted by z_1 and z_2 respectively, but it is convenient to let z_1 and z_2 represent magnitudes only, the use of the symbol j limiting the case under consideration to unreal impedances as in a filter and the sign of z_1 is made opposite to z_2 as must be the case in the transmitting range. r_1 and r_2 are the "dissipation constants" of the series and shunt elements respectively, that is, the resistances associated with z_1 and z_2 are r_1z_1 and r_2z_2 respectively—see diagram (*a*).



(*a*)

Let I_1 be the current flowing in z_1 and I_2 that flowing in z_2, then since z_2 and Z_{02} are in parallel:—

$$I_2 = I_1 \times \frac{\dfrac{z_1z_2}{\sqrt{z_1z_2 - z_1^2}}}{\dfrac{z_1z_2}{\sqrt{z_1z_2 - z_1^2}} - jz_2}$$

No account of the resistances r_1z_1 and r_2z_2 has been taken, but if these are very small their effect on the distribution of currents in the network will be negligible.

Being concerned only with the magnitude of I_2 the equation may be written in this way:—

$$I_2 = I_1 \times \frac{z_1z_2}{\sqrt{(z_1z_2)^2 + z_2^2(z_1z_2 - z_1^2)}} = I_1 \times \sqrt{\frac{z_1}{z_2}}$$

Neglecting for the moment the resistances of the series and shunt elements we may say the power input to the half-section = $I_1^2 Z_{01}$. The power dissipated in the series and shunt arms is given by the expression :—

$$I_1^2 r_1 z_1 + I_2^2 r_2 z_2 = I_1^2 r_1 z_1 + I_1^2 \times \frac{z_1}{z_2} \times r_2 z_2$$
$$= I_1^2 z_1 (r_1 + r_2)$$

Without serious error we may state that the total power input to the half-section is :—

$$I_1^2 Z_{01} + I_1^2 z_1 (r_1 + r_2)$$

$$\frac{\text{Power input}}{\text{Power output}} = \frac{\text{Power input}}{\text{Power input—Losses}}$$

$$= \frac{I_1^2 Z_{01} + I_1^2 z_1 (r_1 + r_2)}{I_1^2 Z_{01}} = 1 + \frac{z_1 (r_1 + r_2)}{Z_{01}}$$

The power ratio for a full section would be :—

$$\left[1 + \frac{z_1 (r_1 + r_2)}{Z_{01}} \right]^2$$

The attenuation constant (Nepers) per section then would be :—

$$\beta = \log_e \left[1 + \frac{z_1 (r_1 + r_2)}{Z_{01}} \right]$$

Since $\frac{z_1 (r_1 + r_2)}{Z_{01}}$ will be small we may write :—

$$\beta = \frac{z_1 (r_1 + r_2)}{Z_{01}} \qquad \text{(1)}$$

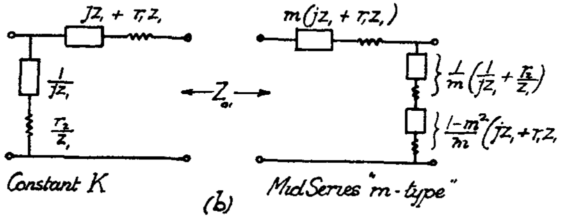
If applied to the case of the simple low-pass filter the expression becomes :—

$$\beta = \frac{\frac{\omega}{\omega_0} (r_1 + r_2)}{\sqrt{1 - \left(\frac{\omega}{\omega_0}\right)^2}} \qquad \text{(2)}$$

In the discussion which follows r_1 and r_2 will stand for the dissipation constants of the elements of a constant K type filter section.

Using the notation of the latter part of Appendix I. we have that $z_2 = \frac{1}{z_1}$

Consider a half-section of mid-series derived m-type as shown in diagram (a).



(b)

It has been assumed that the elements of the series and shunt arms are composed of the same quality inductances and condensers as those of the parent filter.

The dissipation constant of the series arm remains unchanged $= r_1$.

The dissipation constant of the shunt element is :—

$$\frac{\dfrac{r_2}{z_1} + (1 - m^2)r_1z_1}{\dfrac{1}{z_1} - (1 - m^2)z_1} = \frac{r_2 + (1 - m^2)r_1z_1^2}{1 - (1 - m^2)z_1^2}$$

The sum of the two dissipation constants reduces to

$$\frac{r_1 + r_2}{1 - (1 - m^2)z_1^2}$$

Substituting this expression in place of $r_1 + r_2$ and writing mz_1 for z_1 in equation (1) we obtain this equation :—

$$\beta_m = \frac{r_1 + r_2}{1 - (1 - m^2)z_1^2} \times \frac{mz_1}{Z_{01}} \quad \text{.......................(3)}$$

This result holds good also in the case of mid-shunt derived sections. The equation is of general application for any family of " m-type " sections which are derived from a filter in which $z_1z_2 = K^2$.

In the case of low-pass filters this equation can be written :—

$$\beta_m = \frac{r_1 + r_2}{1 - (1 - m^2)\left(\dfrac{\omega}{\omega_0}\right)^2} \times \frac{m\dfrac{\omega}{\omega_0}}{\sqrt{1 - \left(\dfrac{\omega}{\omega_0}\right)^2}} \quad (4)$$

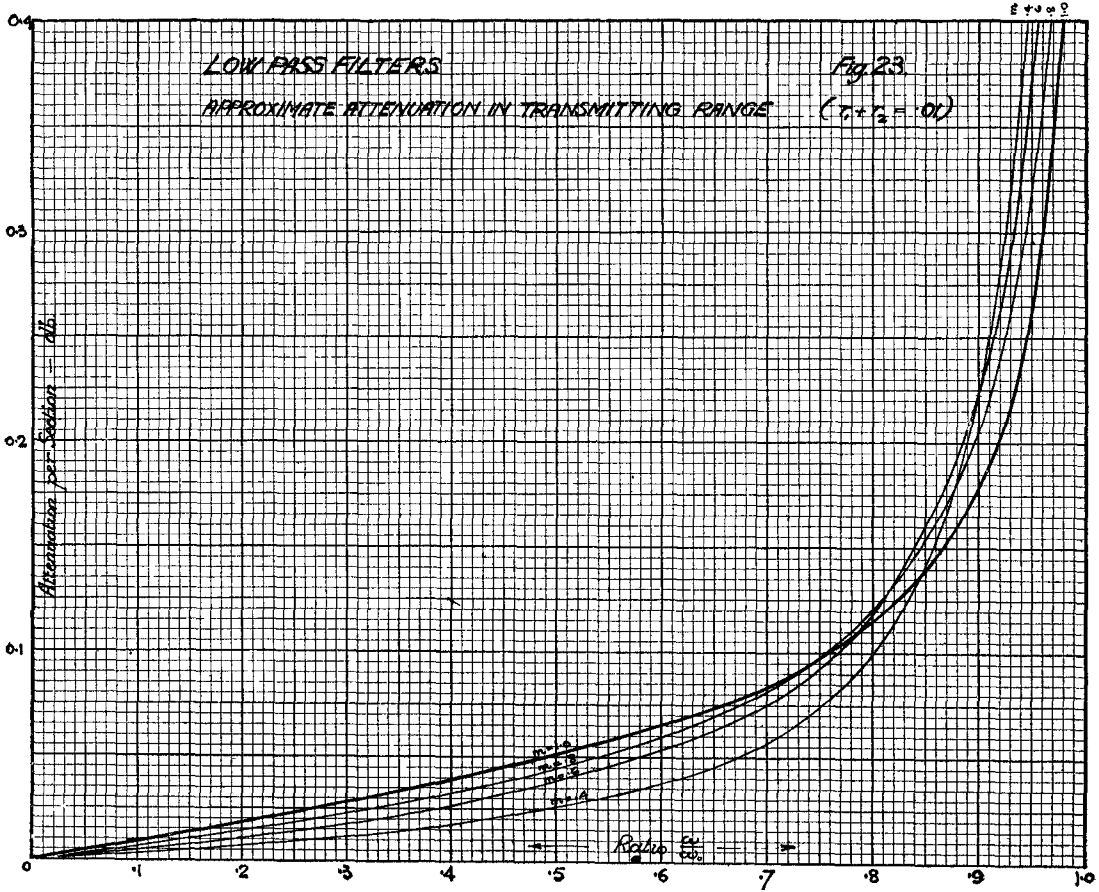


FIG. 23.

If the attenuation of the parent low-pass filter has been calculated, that of a derived type is obtainable by multiplication by the factor

$$\frac{m}{1 - (1 - m^2)\left(\frac{\omega}{\omega_0}\right)^2}$$

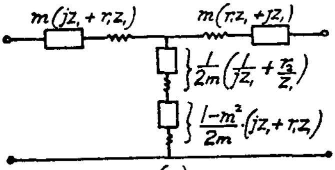
The attenuations are in Nepers—the conversion factor for expression in decibels is 8.69. The corresponding equation to (4) for high-pass filters will be obtained by writing $\frac{\omega_0}{\omega}$ in place of $\frac{\omega}{\omega_0}$.

As the resistances of the elements of a filter are reduced and approach the limit zero the approximate expressions which have been obtained will become exact. Curves for the family of derived low-pass filters are given in Fig. 23 for the case of $r_1 + r_2 = .01$. The attenuation for other values of $r_1 + r_2$ can be taken in direct proportion. Attenuations estimated in this way will be too high, but the error in cases in which elements with good angles are concerned will be small.

APPENDIX IV.

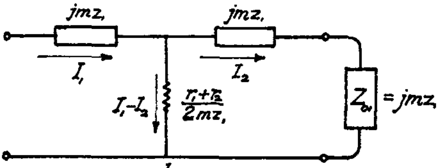
Attenuation at Critical Frequency.

Consider a complete mid-series derived section of a low-pass filter, shown at (a) and using the same symbols as in Appendix III. :—



(a)

At the critical frequency, at which maximum attenuation occurs, the reactances in the shunt arm are equal, but of opposite sign, the reactances cancelling each other out and leaving only the resistances. Neglecting resistances in the series arms, the diagram for the critical frequency becomes :—



(b)

Since the resistance of the shunt arm is small we may write without appreciable error

$$|e^{\gamma}| = \frac{I_1}{I_2} = \frac{4m^2 z_1^{\,2}}{r_1 + r_2}$$

since this is a low-pass filter, $|e^{\gamma}| = \dfrac{4m^2 \left(\dfrac{\omega}{\omega_0}\right)^2}{r_1 + r_2}$

(equation (17)—Appendix I.)
but at critical frequency

$$\frac{\omega_1}{\omega_0} = \frac{1}{\sqrt{1 - m^2}} \left[\begin{array}{l} \text{Found by equating :—} \\ \dfrac{1}{2m}\,\dfrac{1}{z_1} \text{ and } \dfrac{1 - m^2}{2m}\, z_1 \end{array} \right]$$

we may therefore write $|e^{\gamma}| = \dfrac{4m^2}{(r_1 + r_2)(1 - m^2)}$

The attenuation, (in Nepers), per section is :—

$$\log_e \frac{4m^2}{(r_1 + r_2)(1 - m^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If the attenuation is required in decibels the common log of the ratio is to be multiplied by 20.

The mid-shunt characteristic impedance is, by inspection

$$\frac{r_1 + r_2}{mz_1} = \frac{r_1 + r_2}{m \frac{\omega_1}{\omega_0}} = \frac{(r_1 + r_2)\sqrt{1 - m^2}}{m}$$

or rather, since we have taken K = 1 we should express it :—

$$\text{at } \omega_1 \; Z_{02m} = K\,(r_1 + r_2)\,\frac{\sqrt{1 - m^2}}{m} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

The foregoing results are correct for the mid-shunt m-type section except that the expression for the mid-series impedance at the critical frequency is :—

$$\text{at } \omega_1 \; Z_{01m} = \frac{Km}{(r_1 + r_2)\sqrt{1 - m^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Equations (1), (2) and (3) are applicable to the corresponding high-pass filter sections without alteration.

When it is necessary to take account of the impedances given by equations (2) or (3) in calculating terminal loss it is to be noted that these impedances are non-reactive.

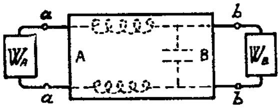
APPENDIX V.

(Added subsequent to the reading of the paper).

Image Impedance and Characteristic Impedance.

The term "characteristic impedance" as originally defined applies strictly only to lines whose electrical constants are uniformly distributed, and American writers use the terms "image impedance" and "iterative impedance" for analogous impedances when referring to filters and other networks. In the present paper the term "characteristic impedance" has been used in reference to loaded lines and filters in the sense for which the term image impedance was invented. Definitions of image impedance and iterative impedance follow:—

Image Impedance.

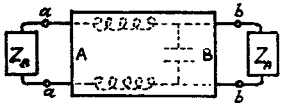


(a)

Let diagram (*a*) represent any network with pairs of input and output terminals, *aa, bb.*

The network is shown terminated at ends A and B by the image impedances W_A and W_B appropriate to the respective ends. The two image impedances are such that the impedance measured between terminals *aa* (looking towards the right), W_a being disconnected, is equal to the image impedance W_a. Similarly the impedance between terminals *bb* (looking to the left), W_b being disconnected, is equal to the image impedance W_b.

Iterative Impedance.

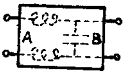


(b)

Diagram (*b*) represents the network terminated by the appropriate iterative impedances Z_B and Z_A at A and B respectively. Z_A and Z_B are such that the impedance between terminals *aa* (looking to the right), Z_B being disconnected, is equal to the iterative impedance Z_A (end A). Similarly the iterative impedance Z_B (end B), is equal to the impedance between terminals *bb*, Z_A being disconnected.

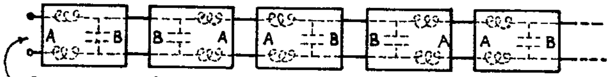
In the case of unsymmetrical networks, as for instance the half low-pass filter section shown by broken lines within the rectangles of diagrams (*a*) and (*b*) the image and iterative impedances at the two ends of the network are different. In the case of symmetrical networks, as for instance a complete filter section, the image impedances at the two ends of the network are equal. The iterative impedances at the two ends are also equal and there is no distinction between iterative and image impedance.

The term "characteristic impedance" is in general use amongst telephone engineers in this country in reference to loaded lines. Since image impedance and iterative impedance are identical in the case of any symmetrical network the retention of the term characteristic impedance in reference to loaded lines and networks could only give rise to ambiguity in the case of unsymmetrical structures. The iterative impedance of an unsymmetrical network appears to have little connection with present-day views of transmission through loaded lines and filters, though it is, no doubt, of mathematical interest. It is suggested that "characteristic impedance," when used in reference to networks, (including loaded lines and filters) should be understood in the same sense as image impedance as defined above.



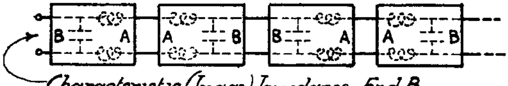
(*c*)

The diagram at (*c*) represents any network AB. Let a series of such networks, identical with AB, be connected in chain as shown at (*d*), alternate networks being reversed in direction. If this series is continued indefinitely to the right the impedance measured between terminals *aa* at the left hand side of the diagram is the characteristic (image) impedance, end A, of the network AB. Similarly the characteristic impedance of the network, at the end B, is the impedance measured between terminals *bb* at the left hand side of diagram (*e*). Reference should be made to page 12 and Fig. 5.



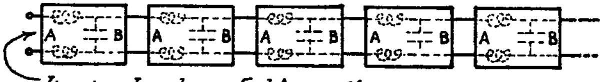
Characteristic (Image) Impedance, End A.

(d)

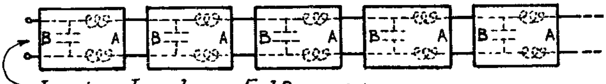


Characteristic (Image) Impedance, End B.

(e)



Iterative Impedance, End A. *(f)*



Iterative Impedance, End B. *(g)*

Diagram (*f*) shows a series of networks connected in chain, but without the reversals of direction shown in diagrams (*d*) and (*f*). If the series is continued indefinitely to the right the impedance measured between terminals *aa* on the left hand side will be iterative impedance at end A of the network AB. The corresponding diagram illustrating the iterative impedance at the end B is shown at (*g*).

APPENDIX VI.

(Added subsequently to the reading of the paper).

When alternating current which is being transmitted along a line has reached a steady state, positions at which the instantaneous values of the current (or voltage) are zero progress along the line with a speed which is equal to $\frac{\omega}{a}$ miles per second, where

a = wave-length constant expressed in radians per mile,

$\omega = 2\pi f$, f being the frequency in p.p.s. of the current.

In the transmission of speech, however, it is the speed of propagation of changes in amplitude that is of first importance. The speed of propagation of changes in amplitude is dependent upon the slope of the wave-length constant-frequency characteristic. This speed of propagation is equal to $\frac{1}{\frac{da}{d\omega}}$ and the time of transmission per mile is $\frac{da}{d\omega}$

Equation (14) of Appendix I. states :—

$$\cosh \gamma_m = 1 + \frac{2m^2 z_1^{\,2}}{1 + (1 - m^2) z_1^{\,2}}$$

and in the transmitting range of an ideal filter

$$\cos a_m = 1 - \frac{2m^2 |z_1|^2}{1 - (1 - m^2)|z_1|^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where a_m is the angular change of phase (expressed in radians) per section. a_m corresponds to the wave-length constant of a line. z_1 is impedance of the series element of the parent (constant k) filter. Differentiating with respect to z_1 :—

$$\frac{da_m}{d|z_1|} = \frac{m}{1 - (1 - m^2) z_1^{\,2}} \times \frac{2}{\sqrt{1 - |z_1|^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Applying this equation to low-pass filters (See end of Appendix I.) :—

$$\frac{da_m}{d\left(\frac{\omega}{\omega_0}\right)} = \frac{m}{1 - (1 - m^2)\left(\frac{\omega}{\omega_0}\right)^2} \times \frac{2}{\sqrt{1 - \left(\frac{\omega}{\omega_0}\right)^2}} \quad \ldots\ldots (3)$$

To obtain the time of transmission or delay per section it is necessary to divide the expression by ω_0.

Fig. 24 gives curves of $\dfrac{d a_m}{d\left(\dfrac{\omega}{\omega_0}\right)}$ for a series of values of m.

The curve for $m = 1$ is applicable to loaded lines except at very low frequencies and frequencies in the neighbourhood of the cut off frequency. Owing to the crowding together and crossing of the curves in Fig. 24 it is difficult to use in the region above $\dfrac{\omega_0}{\omega} = \cdot 8$, but the values from which the curves were plotted are given in Table I. :—

Table I.

$\dfrac{da}{d\left(\dfrac{\omega}{\omega_0}\right)}$ FOR LOW-PASS FILTERS.

$\dfrac{\omega}{\omega_0}$	Values of m.										
	·3	·4	·5	.6	·7	.8	·9	1.0	1.2	1.5	2.0
0	.600	.800	1.000	1.200	1.400	1.60	1.80	2.00	2.40	3.00	4.00
.2	635	845	1.052	1.26	1.46	1.66	1.85	2.04	2.41	2.92	3.65
·4	.766	1.01	1.24	1.46	1.66	1.85	2.03	2.18	2.45	2.73	2.95
.6	1.114	1.43	1.71	1.95	2.14	2.30	2.42	2.50	2.59	2.56	2.40
.8	2.39	2.89	3.20	3 39	3.46	3.46	3.42	3.33	3.12	2.78	2.28
·9	5.24	5.74	5.85	5.72	5.48	5.18	4.88	4.59	4.06	3.42	2.67
·93	7.67	7.96	7.74	7.32	6.82	6.32	5.86	5.44	4.73	3.92	3.03
·95	10.74	10.29	9.93	9.12	8.31	7.60	6.97	6.41	5.50	4.52	3.46
·97								8.22	6.98	5.67	4.30
98									8.46	6.85	5.18

Corresponding values of $\dfrac{da}{d\left(\dfrac{\omega}{\omega_0}\right)}$ for high-pass filter sections are given in Table II., tabulated against $\left(\dfrac{\omega_0}{\omega}\right)$.

The equation corresponding to equation (2) which is applicable in the high-pass case is :—

$$\frac{d a_m}{d\left(\dfrac{\omega}{\omega_0}\right)} = \frac{m}{1 - (1 - m^2)\left(\dfrac{\omega_0}{\omega}\right)^2} \times \frac{2\left(\dfrac{\omega_0}{\omega}\right)^2}{\sqrt{1 - \left(\dfrac{\omega_0}{\omega}\right)^2}}$$

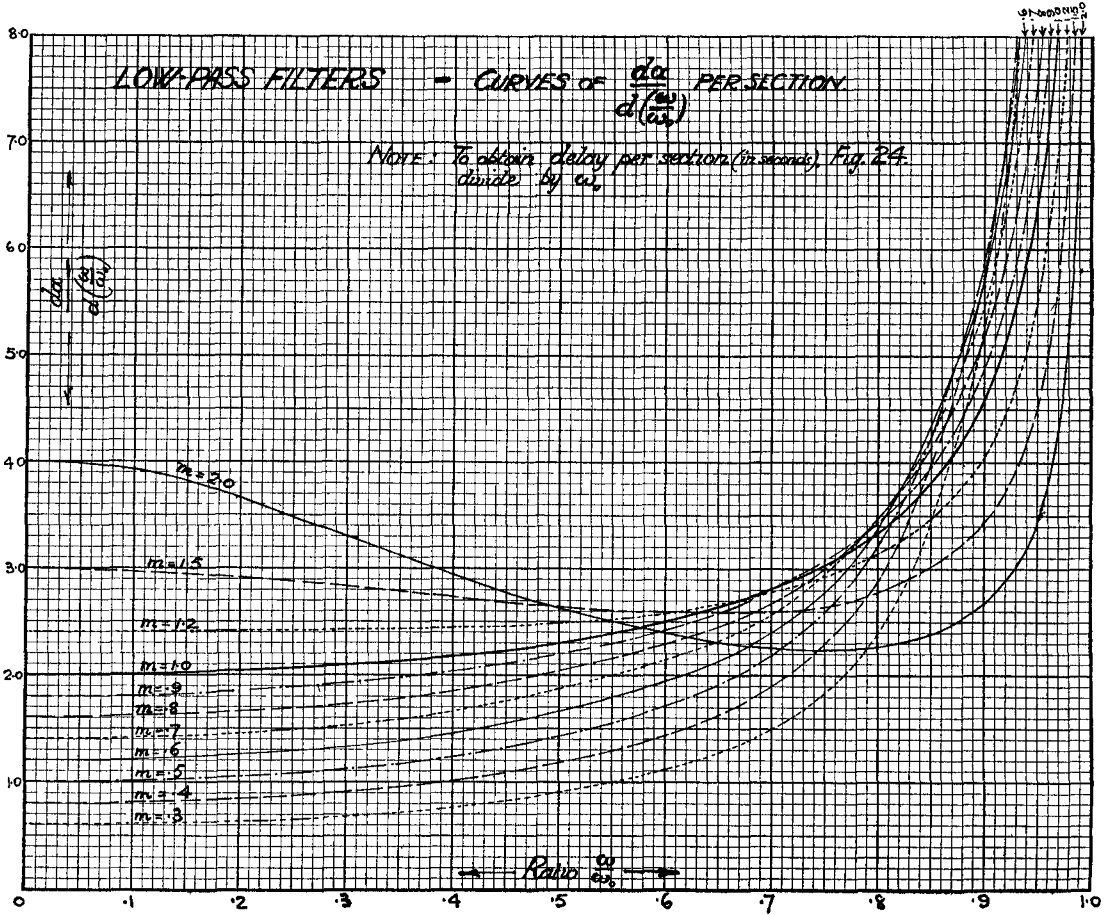

LOW-PASS FILTERS — CURVES OF $\frac{d\alpha}{d(\frac{\omega}{\omega_0})}$ PER SECTION.
NOTE: To obtain delay per section (in seconds), Fig. 24. divide by ω_0
$\frac{d\alpha}{d(\frac{\omega}{\omega_0})}$
Ratio $\frac{\omega}{\omega_0}$
m = 2.0
m = 1.5
m = 1.2
m = 1.0
m = .9
m = .8
m = .7
m = .6
m = .5
m = .4
m = .3
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
·1
·2
·3
·4
·5
·6
·7
·8
·9
1.0

FIG. 24.

Table II.

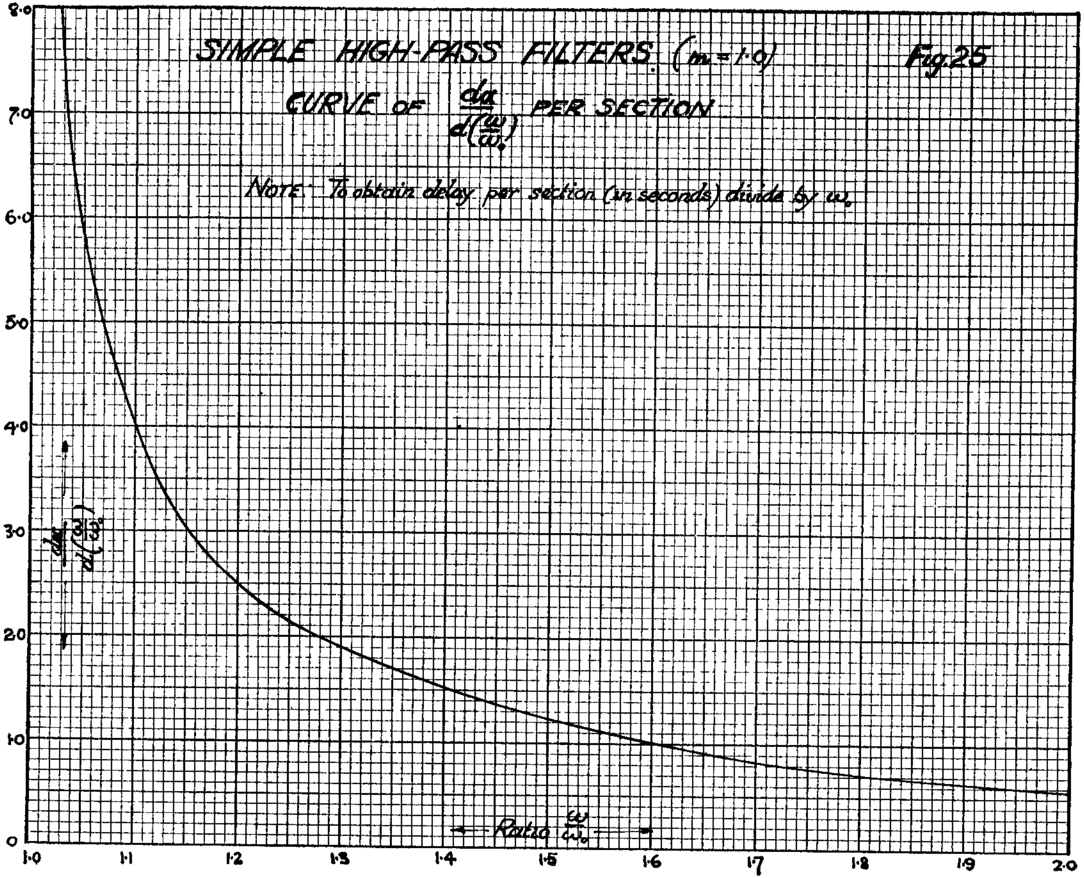
$\dfrac{d\alpha}{d\left(\dfrac{\omega}{\omega_0}\right)}$ FOR HIGH-PASS FILTERS.

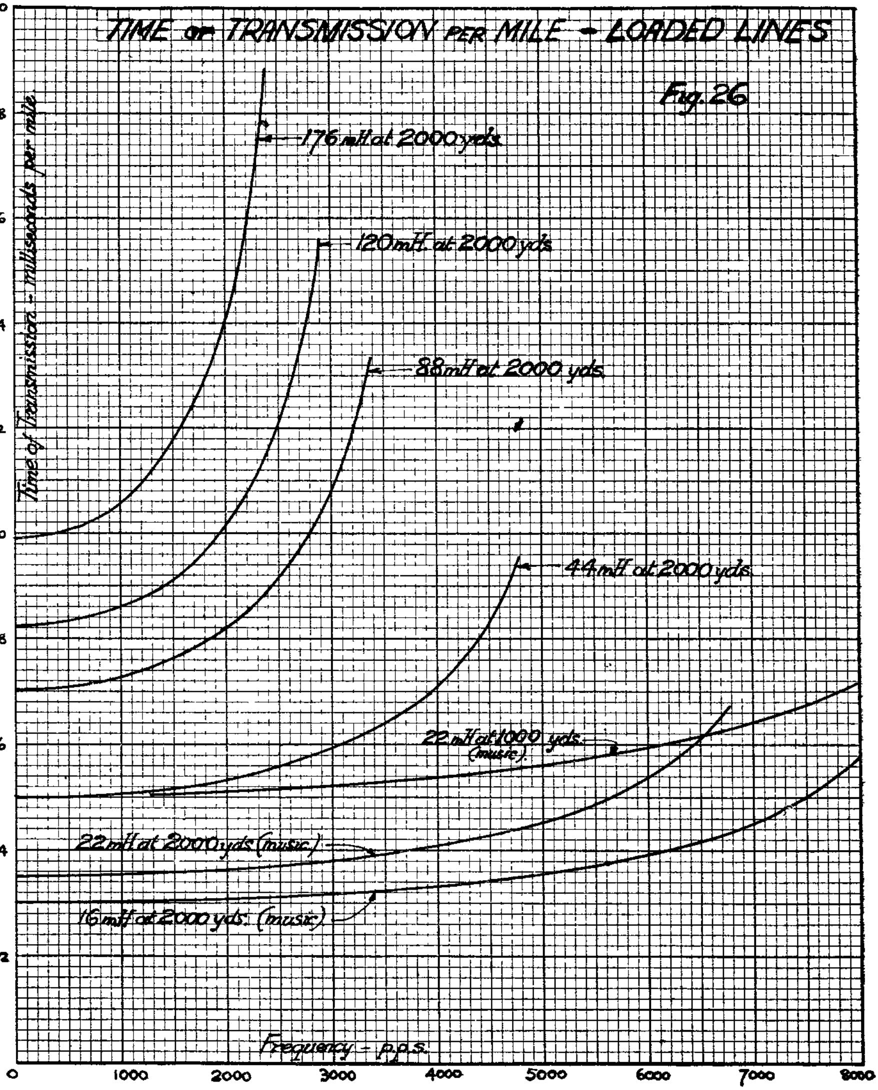
$\dfrac{\omega_0}{\omega}$	Values of *m*.							
	·3	·4	·5	.6	·7	.8	·9	1.0
0	0	0	0	0	0	0	0	0
.2	.025	.034	.042	.050	.058	.066	.075	.082
·4	.123	.161	.198	.233	.266	.296	·329	·349
.6	.401	.516	.616	.701	.770	.829	.870	.900
.8	1.53	1.48	2.05	2.17	2.21	2.22	2.19	2.13
·9	4.24	4.65	4.74	4.63	4.44	4.20	3.95	3.72
·93	6.64	7.68	6.70	6.33	5.90	6.86	5.07	4.70
·95	9.70	9.29	8.96	8.23	7.50	5.47	6.29	5.78

Fig. 25 shows $\dfrac{d\alpha}{d\left(\dfrac{\omega}{\omega_0}\right)}$ plotted against $\dfrac{\omega}{\omega_0}$ for a high-pass filter section of the parent type—*i.e.*, $m = 1.0$.

Curves showing time of transmission plotted against frequency for certain types of loaded line are shown in Fig. 26. Those lines which are used for transmission of music have been indicated by the word " music."

FIG. 25.


SIMPLE HIGH-PASS FILTERS (m=1·0)
Fig.25
CURVE OF $\frac{d\alpha}{d(\frac{\omega}{\omega_0})}$ PER SECTION
NOTE: To obtain delay per section (in seconds) divide by ω_0
$\frac{d\alpha}{d(\frac{\omega}{\omega_0})}$
Ratio $\frac{\omega}{\omega_0}$
8·0
7·0
6·0
5·0
4·0
3·0
2·0
1·0
0
1·0
1·1
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
2·0


TIME OF TRANSMISSION PER MILE - LOADED LINES
Fig. 26
Time of Transmission - milliseconds per mile
176 mH at 2000 yds
120 mH at 2000 yds
88 mH at 2000 yds
44 mH at 2000 yds
22 mH at 1000 yds (music)
22 mH at 2000 yds (music)
16 mH at 2000 yds (music)
Frequency - p.p.s.
·20
·18
·16
·14
·12
·10
·08
·06
·04
·02
0
1000
2000
3000
4000
5000
6000
7000
8000